Modelling (Not so) Thin Films with the Multilayer Solver

P.P. Naanouh[1,2] J.L. Pierson[1] H. Lorcet[1] A. Antkowiak [2]

[1]IFP Energies Nouvelles

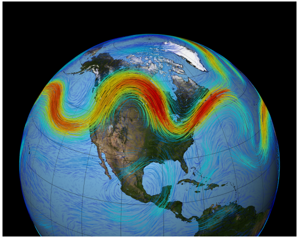
[2]Sorbonne Université

July 8, 2025

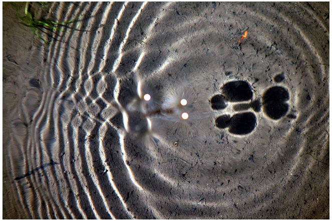
Contents

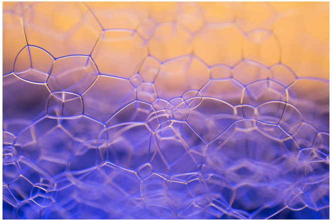
Origins

High aspect ratio flows



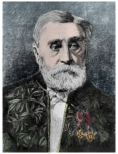
- Atmospheric BL
- Oceanic Flows
- Rivers & Basins
- Films
- Emulsions





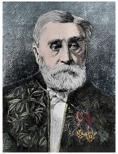
History

- Saint-Venant Model 19th century



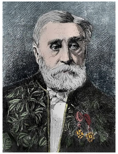
History

- Saint-Venant Model 19th century
  - $\partial_t \eta + \nabla \cdot (\eta \mathbf{u}) = 0$
  - $\partial_t (\eta \mathbf{u}) + \nabla \cdot (h \mathbf{u} \otimes \mathbf{u}) = -\eta g \nabla \eta$



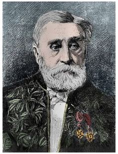
History

- Saint-Venant Model 19th century
  - $\partial_t \eta + \nabla \cdot (\eta \mathbf{u}) = 0$
  - $\partial_t (\eta \mathbf{u}) + \nabla \cdot (h\mathbf{u} \otimes \mathbf{u}) = -\eta g \nabla \eta$
- Extention to multilayer Saint-Venant in later 20th century



History

- Saint-Venant Model 19th century
  - $\partial_t \eta + \nabla \cdot (\eta \mathbf{u}) = 0$
  - $\partial_t (\eta \mathbf{u}) + \nabla \cdot (h \mathbf{u} \otimes \mathbf{u}) = -\eta g \nabla \eta$
- Extention to multilayer Saint-Venant in later 20th century
  - $\partial_t h_k + \nabla \cdot (h_k \mathbf{u}_k) = 0$
  - $\partial_t (h_k \mathbf{u}_k) + \nabla \cdot (h_k \mathbf{u}_k \otimes \mathbf{u}_k) = -h_k g \nabla (\sum h_j)$

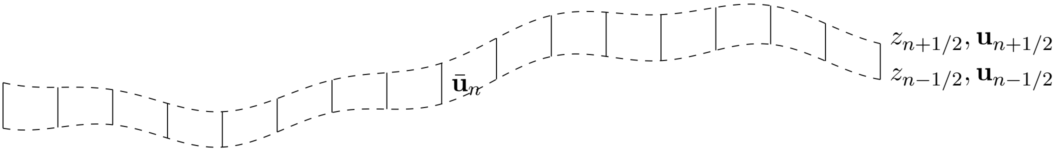


Why do we go further?

- High vertical gradients
- Locally low aspect ratios
- Non-hydrostatic flows
- Unknown term scaling

Multilayer Navier-Stokes model

Characteristics



- Exact reformulation of NS-Equations
- Coordinates are Cartesian
- **Semi-Lagrangian/Integral** (discrete in vertical direction)
- Layers cannot fold
- Layer thickness is directly available
- "Mesh" automatically follows surface shape

Governing Equations

$$\nabla \cdot (h\bar{\mathbf{u}})_k + [w - \mathbf{u} \cdot \nabla \hat{z}]_k = 0; \nabla \cdot (h\bar{\mathbf{u}})_k + \frac{\partial h_k}{\partial t} = 0 \qquad (1)$$

Governing Equations

$$\nabla \cdot (h\bar{\mathbf{u}})_k + [w - \mathbf{u} \cdot \nabla \hat{z}]_k = 0; \nabla \cdot (h\bar{\mathbf{u}})_k + \frac{\partial h_k}{\partial t} = 0 \tag{1}$$

$$\frac{\partial (h\bar{\mathbf{u}})_k}{\partial t} + \nabla \cdot (h\overline{\mathbf{u} \otimes \mathbf{u}})_k = -\nabla \left(h\overline{\phi}\right)_k + [\phi \nabla \hat{z}]_k - \left(g\nabla\eta + \frac{1}{\rho}\nabla\sigma\kappa + A_H \nabla\left(\frac{1}{\eta^3}\right)\right) h_k$$
$$+ \nu \left(\nabla^2 (h\bar{\mathbf{u}})_k - 2[\nabla\mathbf{u} \cdot \nabla\hat{z}]_k - \left[\mathbf{u}\nabla^2 \hat{z}\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z}(\nabla\hat{z} \cdot \nabla\hat{z})\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z}\right]_k\right) \tag{2}$$

Governing Equations

$$\nabla \cdot (h\bar{\mathbf{u}})_k + [w - \mathbf{u} \cdot \nabla \hat{z}]_k = 0; \nabla \cdot (h\bar{\mathbf{u}})_k + \frac{\partial h_k}{\partial t} = 0 \tag{1}$$

$$\frac{\partial (h\bar{\mathbf{u}})_k}{\partial t} + \nabla \cdot (h\overline{\mathbf{u} \otimes \mathbf{u}})_k = -\nabla \left(h\overline{\phi}\right)_k + [\phi \nabla \hat{z}]_k - \left(g\nabla\eta + \frac{1}{\rho}\nabla\sigma\kappa + A_H \nabla \left(\frac{1}{\eta^3}\right)\right) h_k$$
$$+ \nu \left(\nabla^2 (h\bar{\mathbf{u}})_k - 2\left[\nabla \mathbf{u} \cdot \nabla \hat{z}\right]_k - \left[\mathbf{u}\nabla^2 \hat{z}\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z}(\nabla \hat{z} \cdot \nabla \hat{z})\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z}\right]_k\right) \tag{2}$$

$$\frac{\partial (h\bar{w})_k}{\partial t} + \nabla \cdot (h\overline{w\mathbf{u}})_k =$$
$$-[\phi]_k + \nu \left(\nabla^2 (h\bar{w})_k - 2\left[\nabla w \cdot \nabla \hat{z}\right]_k - \left[w\nabla^2 \hat{z}\right]_k + \left[\frac{\partial w}{\partial z}(\nabla \hat{z} \cdot \nabla \hat{z})\right]_k + \left[\frac{\partial w}{\partial z}\right]_k\right) \tag{3}$$

Advection term derivation

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) + \frac{\partial uw}{\partial z} \tag{4}$$

Integration between two mobile surfaces $\hat{z}_{k+1/2}$ and $\hat{z}_{k-1/2}$

$$\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \frac{Du}{Dt} dz = \int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \frac{\partial u}{\partial t} dz + \int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \nabla \cdot (u \otimes u)\, dz + [uw]_{\hat{z}_{k+1/2}}^{\hat{z}_{k+1/2}} \tag{5}$$

Advection term derivation

We apply the Leibniz integral rule to the first two terms

$$\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \frac{\partial u}{\partial t} dz = \frac{\partial}{\partial t}\left(\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} u dz\right) - \left(u_{\hat{z}_{k+1/2}} \frac{\partial \hat{z}_{k+1/2}}{\partial t} - u_{\hat{z}_{k-1/2}} \frac{\partial \hat{z}_{k-1/2}}{\partial t}\right) \qquad (6)$$

$$\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \nabla \cdot (u \otimes u)\, dz = \nabla \cdot \int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} (u \otimes u)\, dz - \left(u_{\hat{z}_{k+1/2}} \left(u_{\hat{z}_{k+1/2}} \cdot \nabla \hat{z}_{k+1/2}\right) - u_{\hat{z}_{k-1/2}} \left(u_{\hat{z}_{k-1/2}} \cdot \nabla \hat{z}_{k-1/2}\right)\right) \qquad (7)$$

Advection term derivation

$$\left[u\frac{\partial z}{\partial t} + u\left(u\cdot\nabla z\right)\right]_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} = \left[u\left(\frac{\partial z}{\partial t} + \left(u\cdot\nabla z\right)\right)\right]_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} = \left[uw\right]_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \tag{8}$$

$$\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \frac{Du}{Dt} dz = \frac{\partial}{\partial t}\left(\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} u dz\right) + \nabla\cdot\int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} \left(u\otimes u\right) dz = \frac{\partial h_k \overline{u}_k}{\partial t} + \nabla\cdot\left(h_k\left(\overline{u\otimes u}\right)_k\right) \tag{9}$$

$\overline{u}_k = \dfrac{1}{\hat{z}_{k+1/2} - \hat{z}_{k-1/2}} \int_{\hat{z}_{k-1/2}}^{\hat{z}_{k+1/2}} u dz$ is the **average** of u between $\hat{z}_{k-1/2}$ and $\hat{z}_{k+1/2}$.

Slope Terms

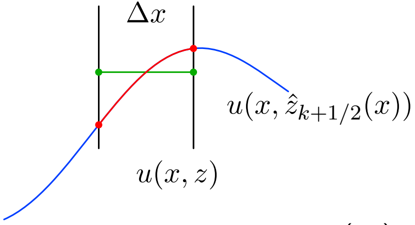
Apply **only** on the layer interfaces either as interlayer coupling terms or boundary conditions Appear in:

1. Diffusion term
2. Surface stresses
3. Continuity equation
4. Pressure term

Slope terms:

Beware $\left.\frac{\partial s(x,z)}{\partial x}\right|_{\hat{z}_{k+1/2}} \neq \frac{\partial s_{k+1/2}}{\partial x} = \frac{\partial s(x, \hat{z}_{k+1/2}(x))}{\partial x}$

$$\frac{\partial s_{k+1/2}}{\partial x} = \left.\frac{\partial s(x,z)}{\partial x}\right|_{\hat{z}_{k+1/2}} + \left.\frac{\partial s}{\partial z}\right|_{\hat{z}_{k+1/2}} \frac{\partial \hat{z}_{k+1/2}}{\partial x}$$



$$\phi_\nu = -2\nu \frac{1 + \left(\frac{\partial \eta}{\partial x}\right)^2}{1 - \left(\frac{\partial \eta}{\partial x}\right)^2} \left.\frac{\partial u}{\partial x}\right|_{top} \tag{10}$$

$$\left.\frac{\partial u}{\partial z}\right|_{top} = -\left.\frac{\partial w}{\partial x}\right|_{top} + 4\frac{\frac{\partial \eta}{\partial x}}{1 - \left(\frac{\partial \eta}{\partial x}\right)^2} \left.\frac{\partial u}{\partial x}\right|_{top} \tag{11}$$

$$\nu \left(\nabla_s^2 \left(h\bar{\mathbf{u}}\right)_k - 2\left[\nabla \mathbf{u} \cdot \nabla \hat{z}\right]_k - \left[\mathbf{u}\nabla^2 \hat{z}\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z} \left(\nabla \hat{z} \cdot \nabla \hat{z}\right)\right]_k + \left[\frac{\partial \mathbf{u}}{\partial z}\right]_k \right) \tag{12}$$

Test Cases

Heat Diffusion

- Solves $\partial_t T = \Delta T$
- Tested on rectangular and sinusoidal domains
- Slope $0 - 52°$
- Temperature set at bathymetry
- Gradients fixed at surface and left & right boundaries

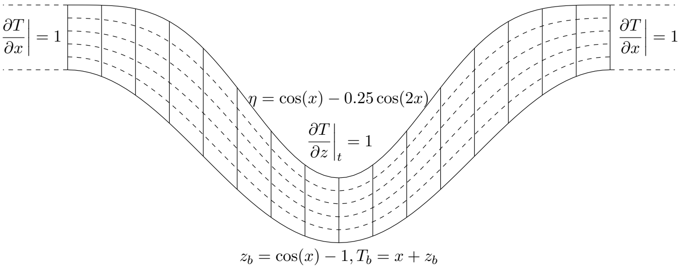


Figure: Computational Domain

Heat Diffusion

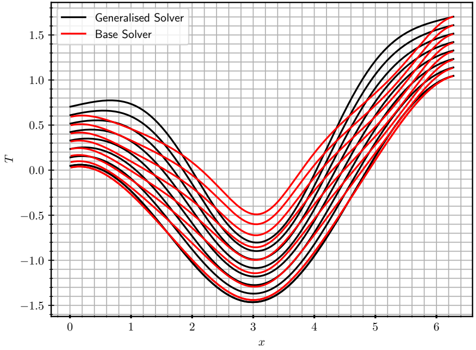


Figure: Temperature Distribution $N = 2^8$

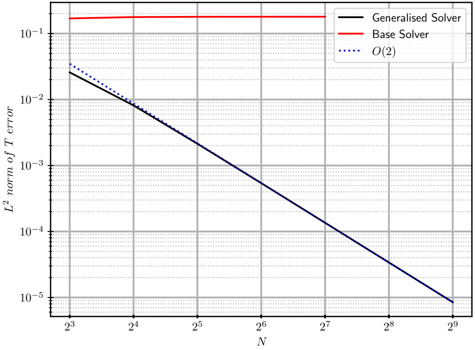


Figure: Mesh Convergence

Plateau Borders

- Form at the intersection of 3 films
- Curvature is constant
- Liquid flows inward
- Can exhibit localized film thinning

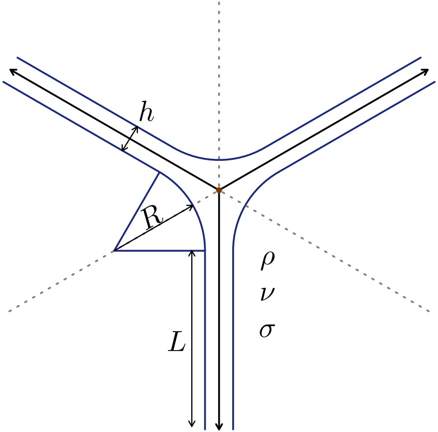


Figure: Plateau border

Plateau Borders

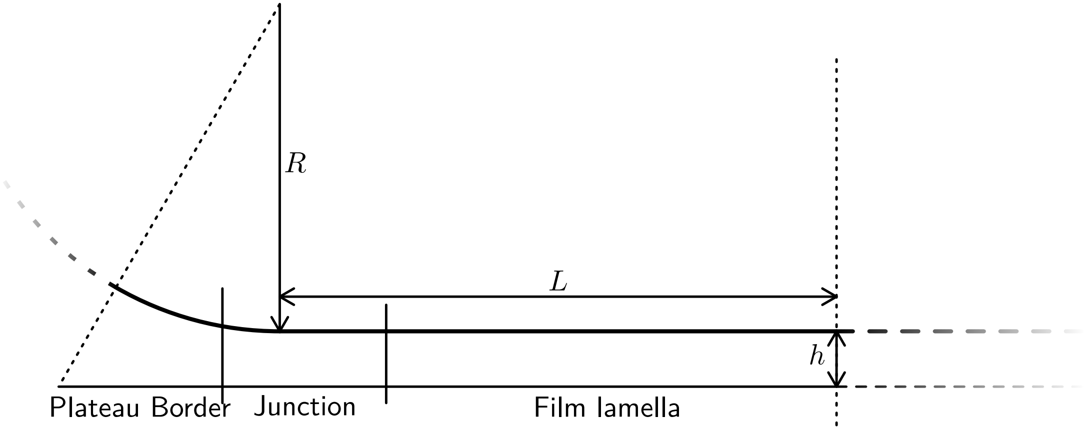


Figure: Plateau border Regions

Plateau Borders

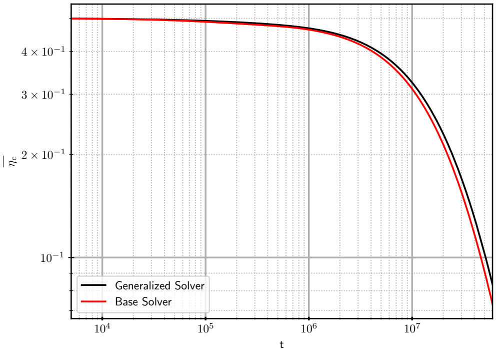


Figure: Pinch thickness

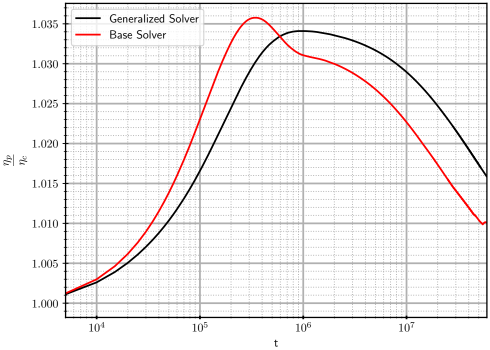


Figure: Centre thickness/Pinch thickness

Plateau Borders

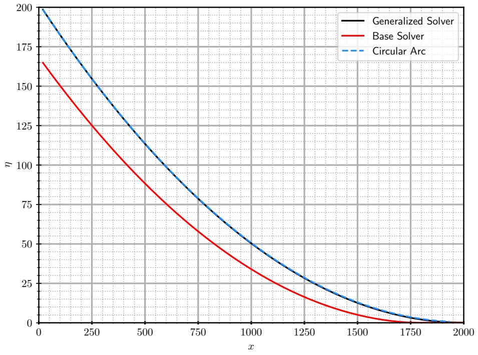


Figure: Surface profile

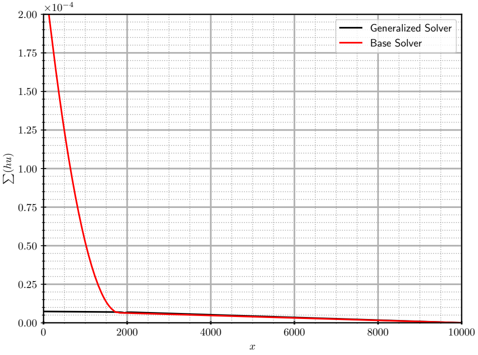


Figure: Flux Profile

Conclusions

Final Thoughts

Conclusions

- System of coupled 2-D equations instead of system of 3-D equations
- All slope terms are implemented and tested

Final Thoughts

Conclusions

- System of coupled 2-D equations instead of system of 3-D equations
- All slope terms are implemented and tested

Perspectives

- Self-similar solution for Plateau Borders
- Implicit diffusion
- Surfactant transport
- Bubble collisions

Thank you!

paul-peter.naanouh@ifpen.fr

http://basilisk.fr/sandbox/pnaanouh/Film_Drainage/