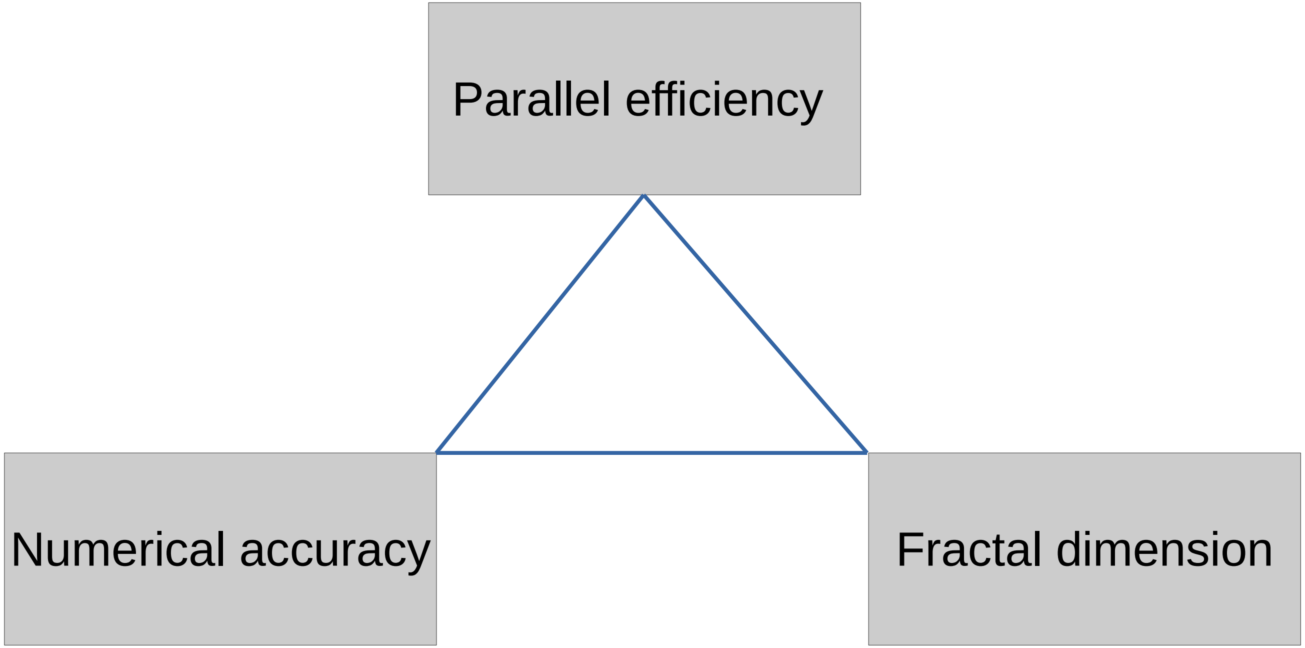
How to Converge on a Tree Grid?

Antoon van Hooft
BGUM 2023, Paris
In collaboration with:
Stephane Popinet

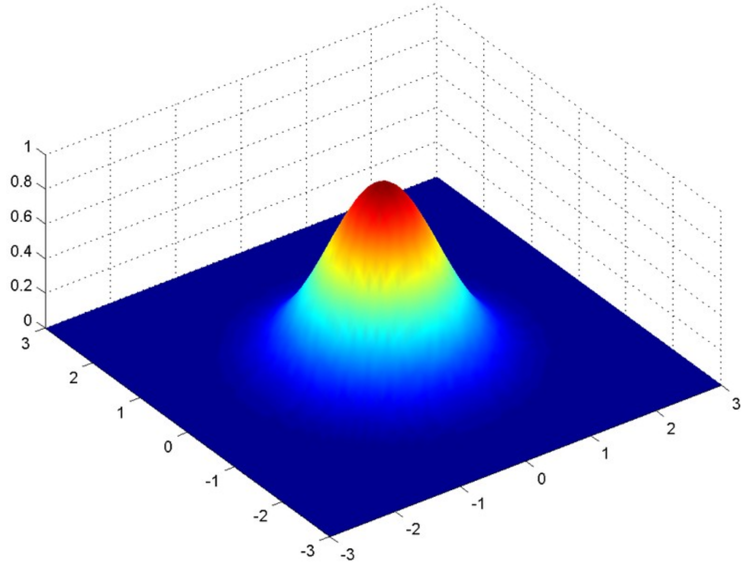
The Weak Scaling Triad of High-Performance Computing



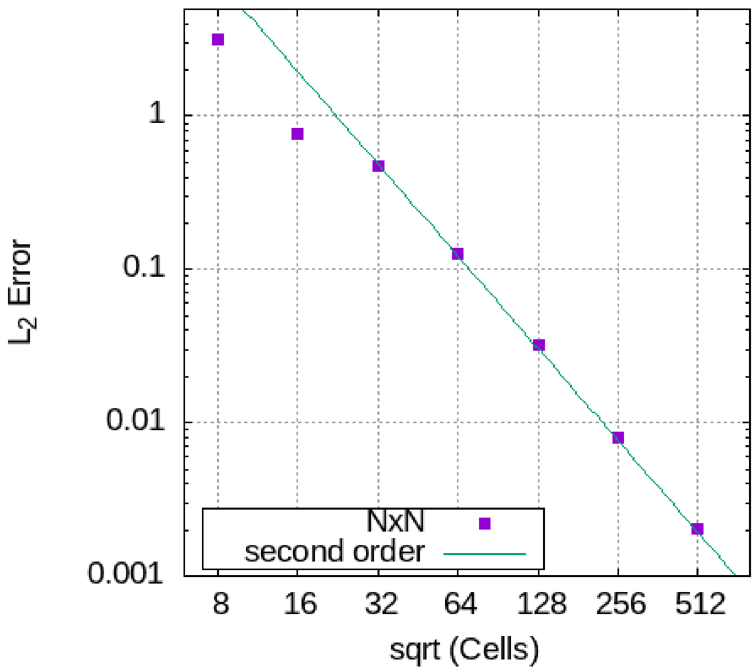
An example

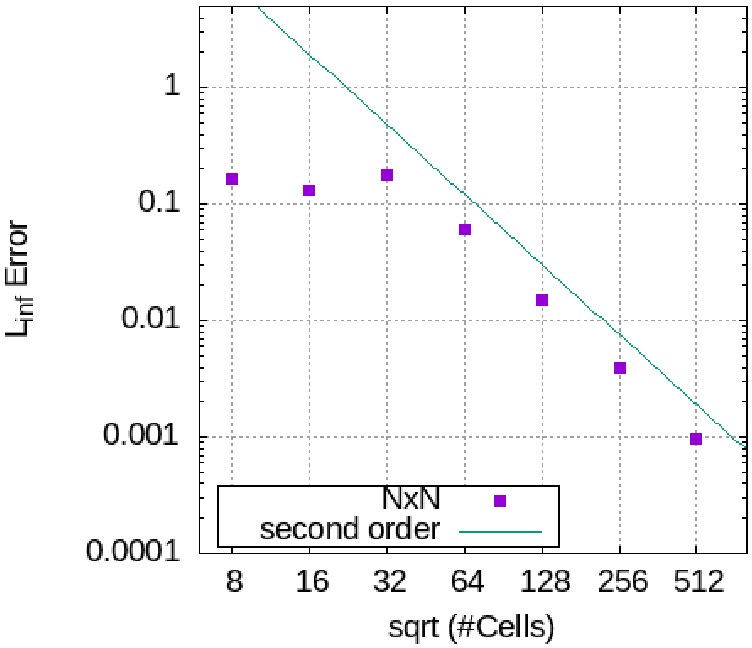
$$\frac{\partial s}{\partial t} = \nabla^2 s$$

$$s(x, y) = e^{-x^2 - y^2}$$



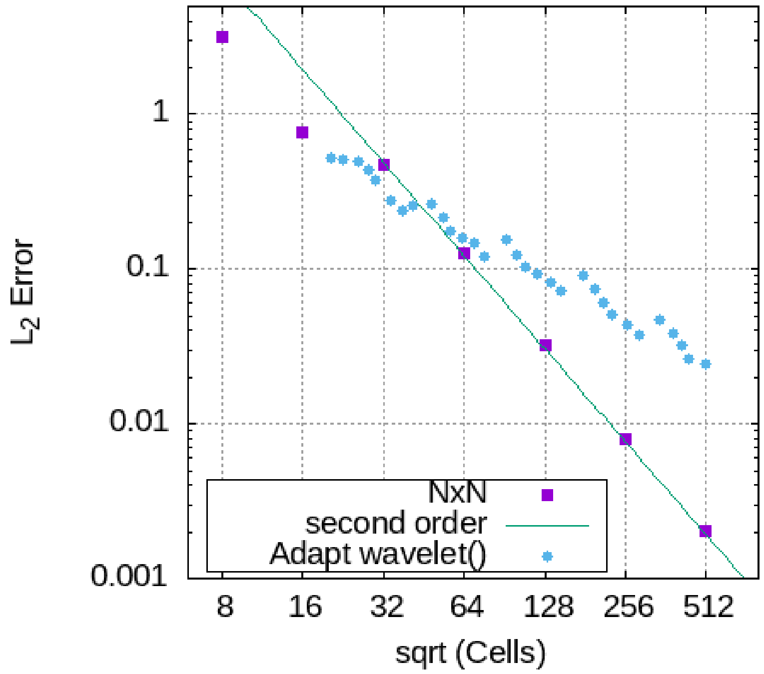
An example; Equidistant grid

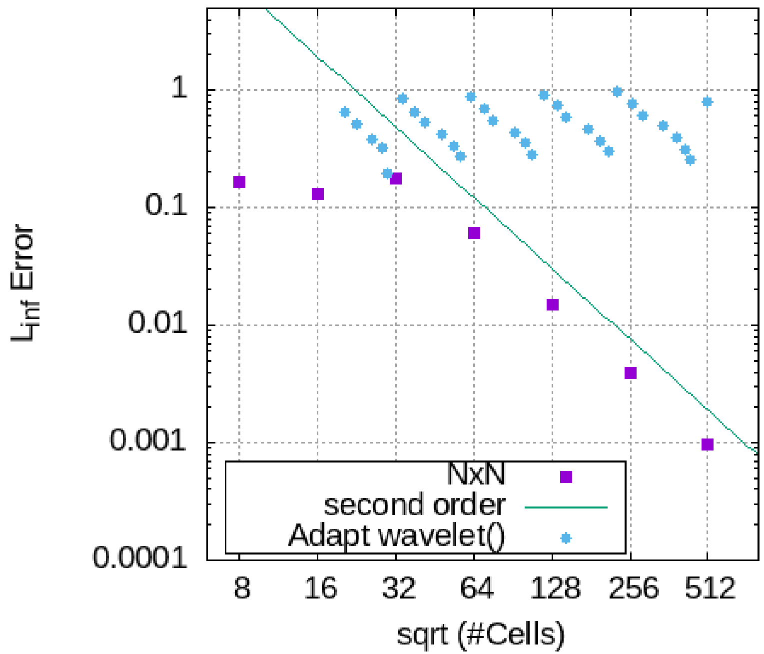




See; www.basilisk.fr/sandbox/antoonvh/diffusion_error.c

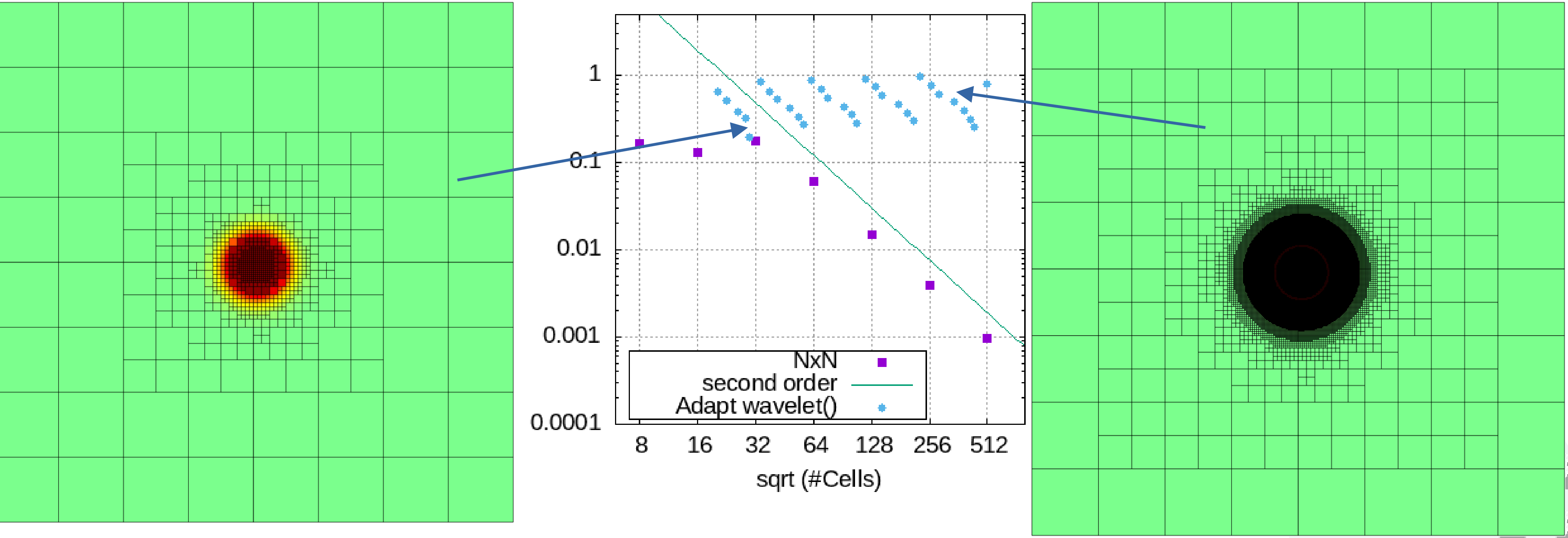
An example; Adaptive grid





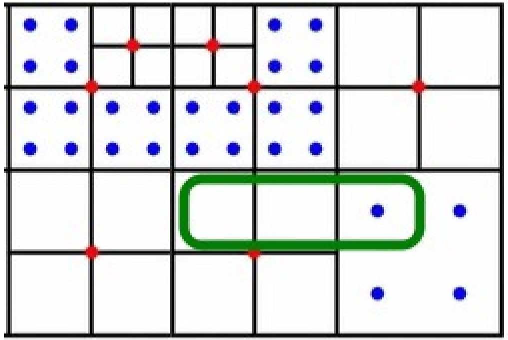
See; www.basilisk.fr/sandbox/antoonvh/diffusion_error.c

An example; Adaptive grid



See; www.basilisk.fr/sandbox/antoonvh/diffusion_error.c

An example; Adaptive grid

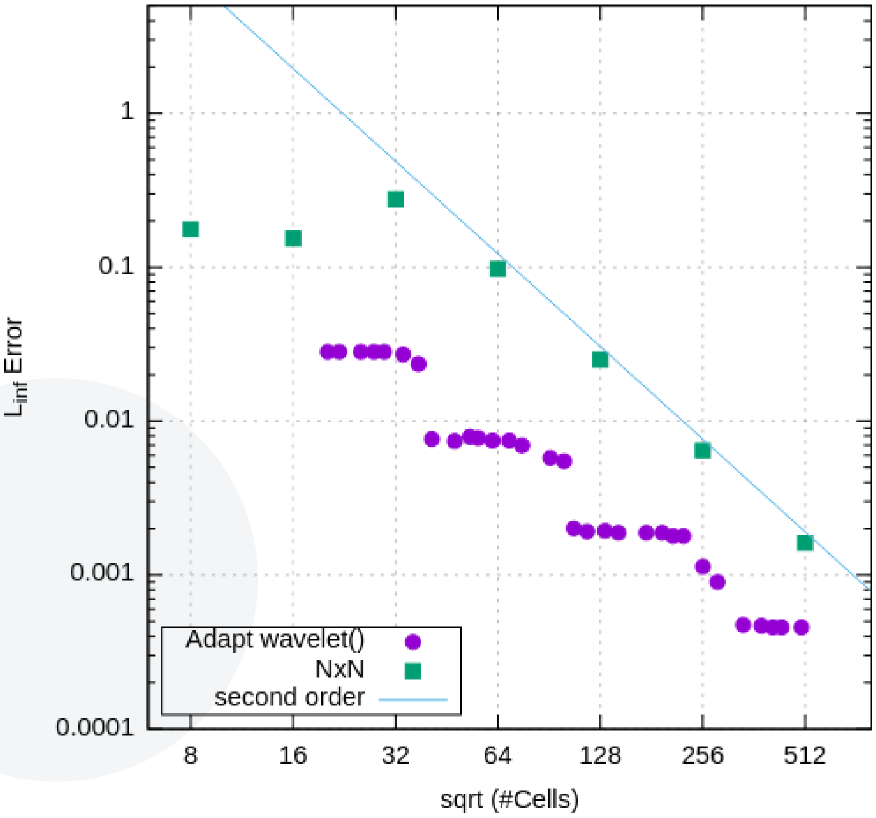


$$\frac{\partial^2 s}{\partial x^2} \approx \frac{s[-1] - 2s[\ ] + s[1]}{\Delta^2} =$$

$$= \frac{s[-1] - 2s[\ ] + s_e[1] + \epsilon\Delta^2}{\Delta^2} =$$

$$= \frac{s[-1] - 2s[\ ] + s_e[1]}{\Delta^2} + \epsilon$$

An example; Adaptive grid



$$s[1] = s_e[1] + \xi\Delta^4$$

```
#include "higher-order.h"
```

```
s.prolongation = refine_4th;
```

See; www.basilisk.fr/sandbox/antoonvh/diffusion_error_fix.c

Higher-order methods?
Recall BGUM 2017...

Sorbonne Université

École Doctorale Sciences Méchaniques, Acoustique, Électronique & Robotique

Institut Jean Le Rond d'Alembert (∂'Alembert UMR7190)

&

Laboratoire d'Informatique de Paris 6 (LIP6 UMR CNRS 7606)

Higher-order adaptive methods for fluid dynamics

By M. Rajarshi Roy Chowdhury

PhD thesis on Fluid Mechanics

Co-directed by M. Stéphane Popinet & M. Stef Graillat

Thesis Defense on 30 November, 2018

Before a jury composed of

Mme. Donna Calhoun	Boise State University, USA	Rapporteuse
M. Frédéric Golay	Université de Toulon, France	Rapporteur
M. Stéphane Vincent	Université Paris-Est-Marne-La-Vallée, France	Examinateur
M. Ivan Delbende	Sorbonne Université, France	Examinateu
M. Stef Graillat	Sorbonne Université, France	
M. Stéphane Popinet	Sorbonne Université, France	

A fourth-order accurate adaptive solver for incompressible flow problems?

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\,\mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{a},$$

$$\nabla \cdot \mathbf{u} = 0.$$

A fourth-order accurate adaptive solver for incompressible flow problems?

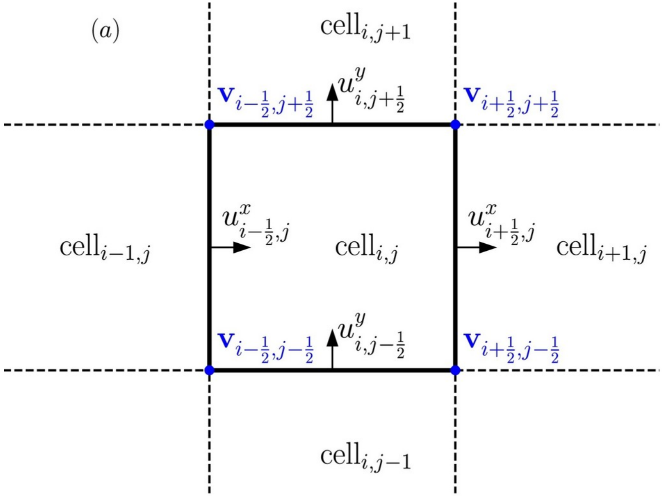
$$\frac{\partial \mathbf{u}}{\partial t} = -\left(\mathbf{u} \cdot \nabla\right)\mathbf{u} - \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{a},$$

$$\nabla \cdot \mathbf{u} = 0.$$

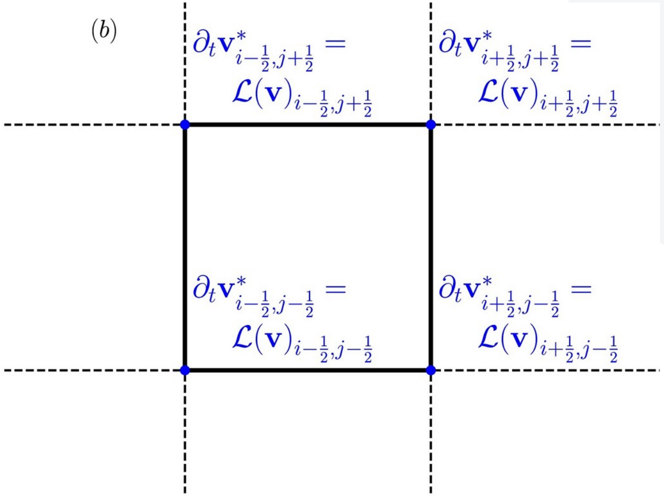
Time integrator:
4th-order accurate
Runge-Kutta scheme

$$\frac{\partial \mathbf{u}}{\partial t} = \mathcal{L}\left(\mathbf{u}\right)$$

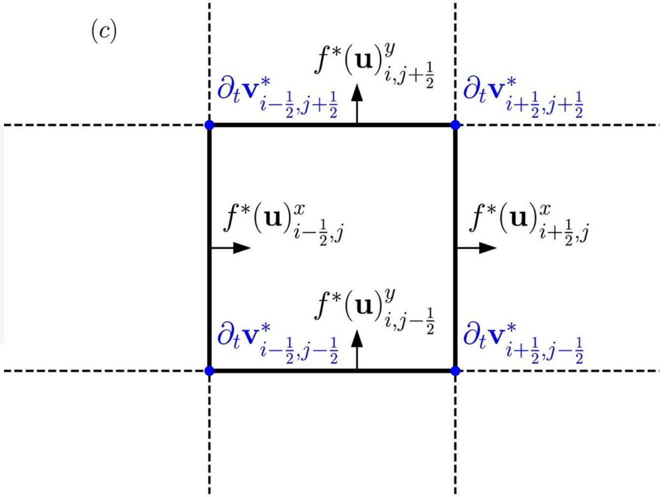
Spatial discretization:
Combined volume-averaged,
Face-averaged and point-value
representation



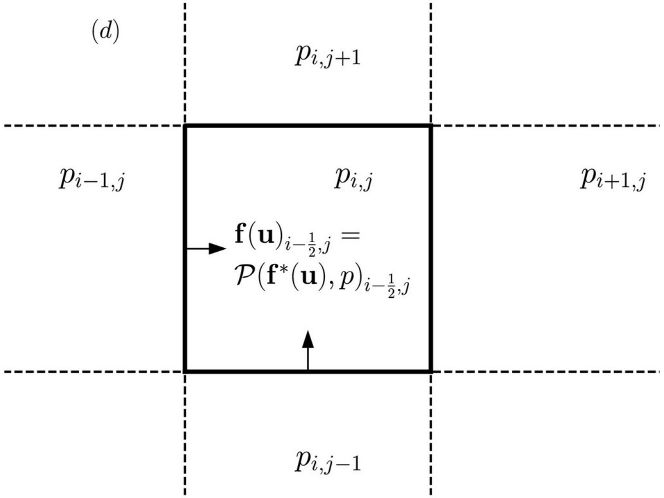
(a) Interpolation of the face-averaged velocities (black) to vertex-point values (blue)



(b) Computation of the provisional tendency vector field at the vertex locations



(c) Interpolation of the provisional tendency from the vertex-point values to face-averaged values.



(d) The (global) solenoidal projection ($\mathcal{P}$) of the provisional tendency using the scalar field p

Advection term requires gradients:

4-th-order accurate central difference:

$$s'[\,] = \frac{1}{12\Delta}\left(s[-2] - 8s[-1] + 8[\,] - s[2]\right) + \epsilon\Delta^4$$

cf. 4-th-order accurate *compact upwind* scheme:

$$s'[\,] - \frac{76}{197}s'[1] + \frac{17}{197}s'[2] =$$

$$\frac{1}{197\Delta}\left(27[-2] - 192s[-1] + 165s[\,]\right) + \epsilon\Delta^4$$

Higher-order adaptive refinement

Refinement indicator

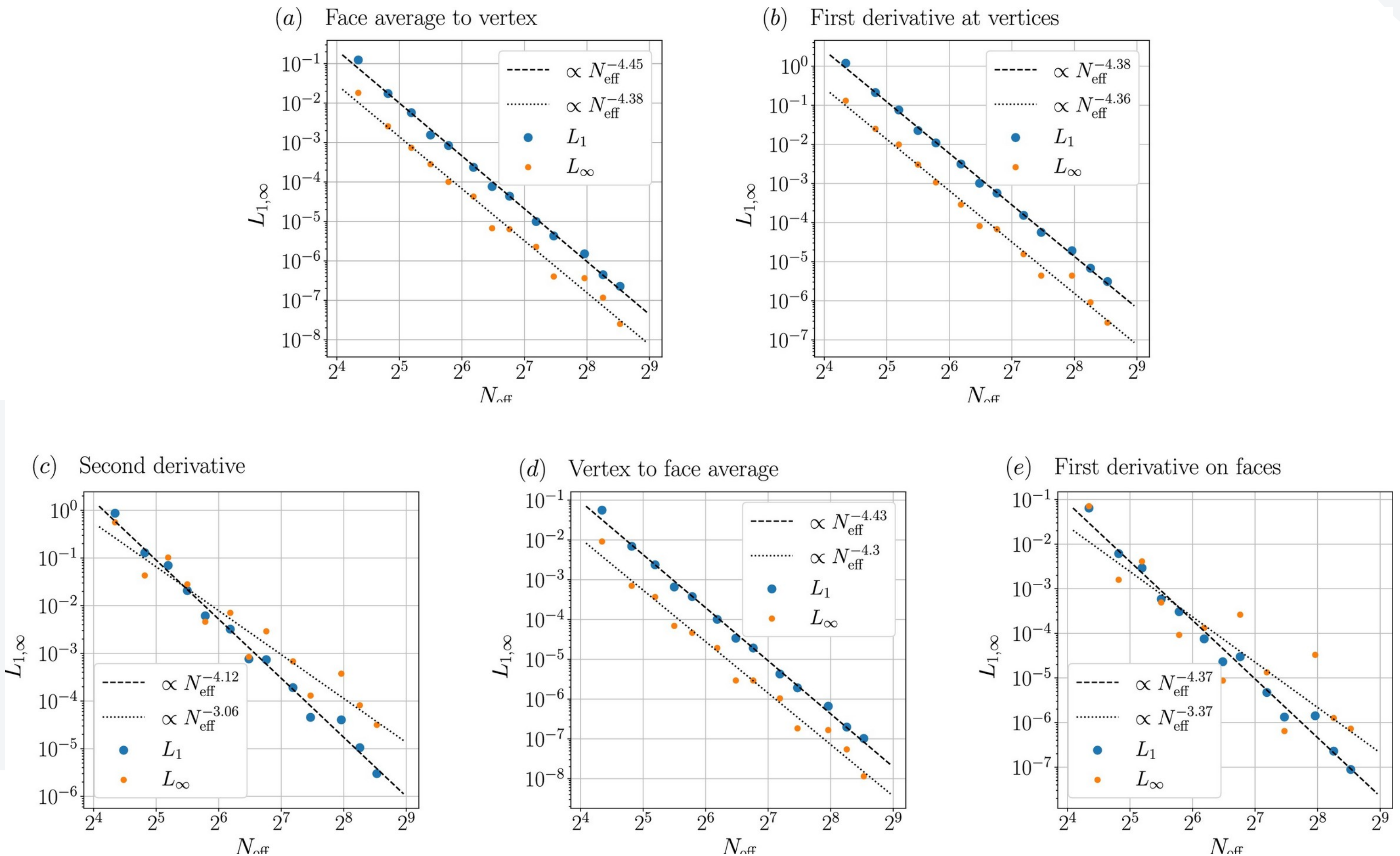
$$\Xi_i = \chi_i$$

Wavelet-based estimated error
i.e. “prolongation error”

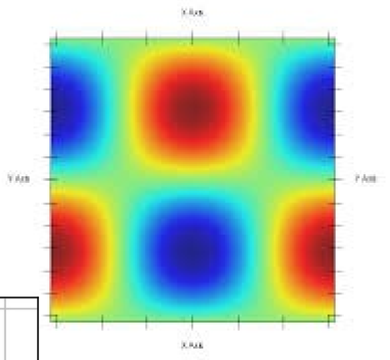
$$\Xi_i = \Delta^k \chi_{\mathrm{i,max\ of\ siblings}}$$

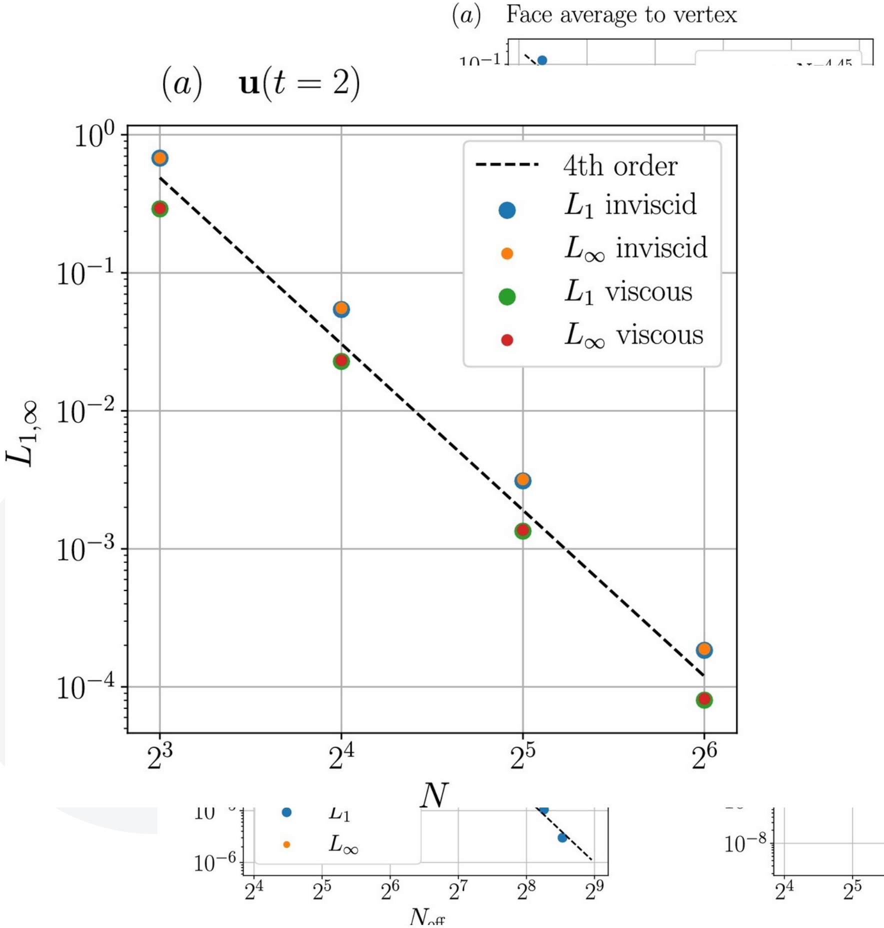
Refinement “penalty” (k > 0)

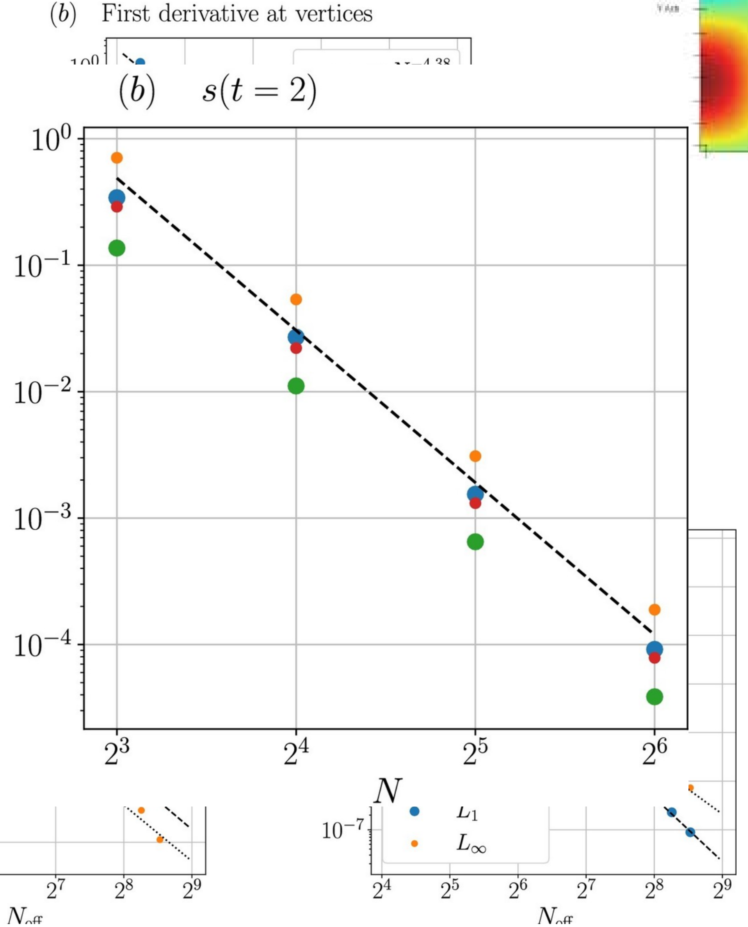
Test: 1) Adaptive convergence for a Gaussian vortex



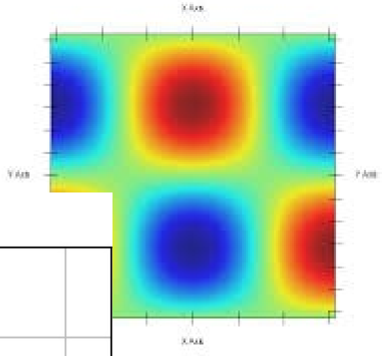
Test: 2) Taylor-Green vortex

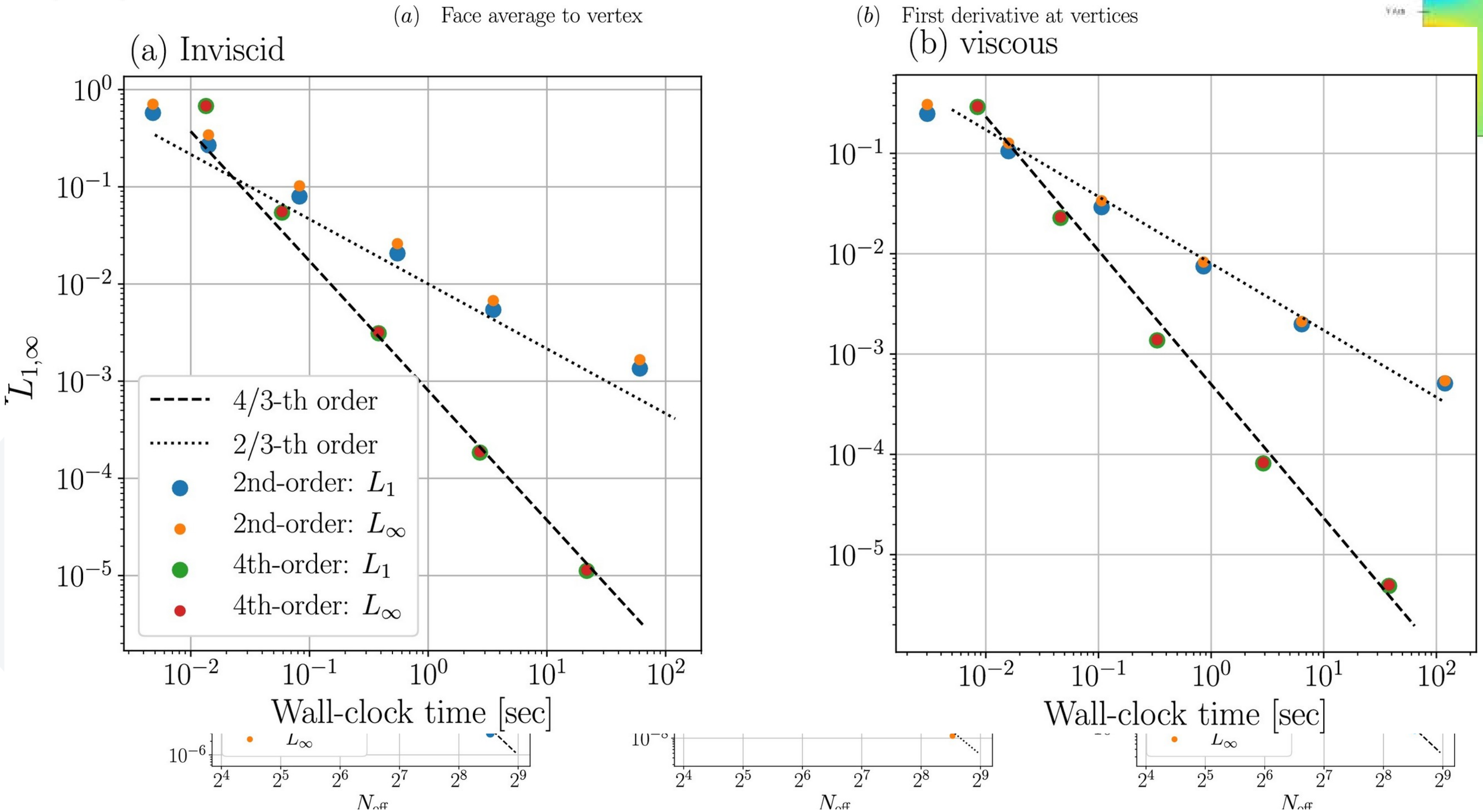




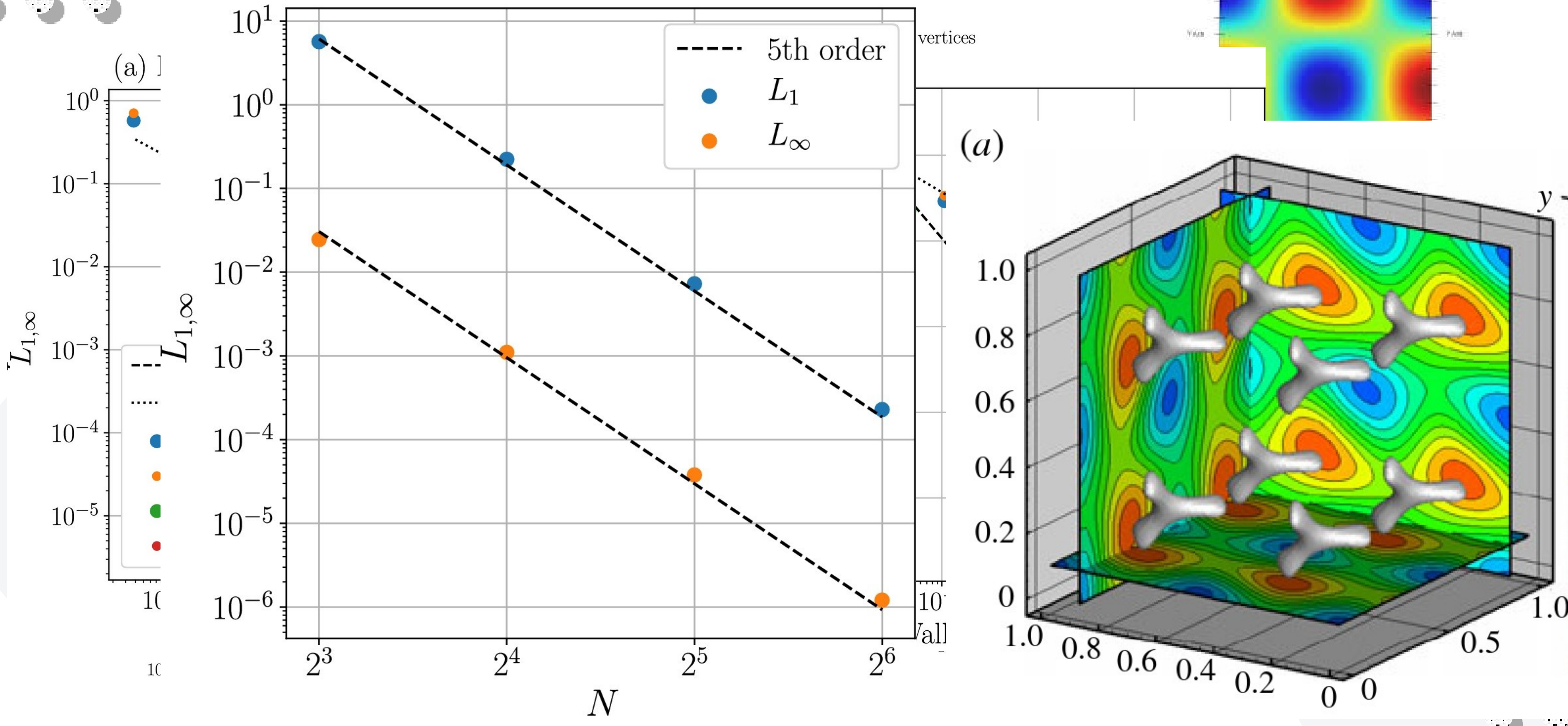


Test: 2) Taylor-Green vortex

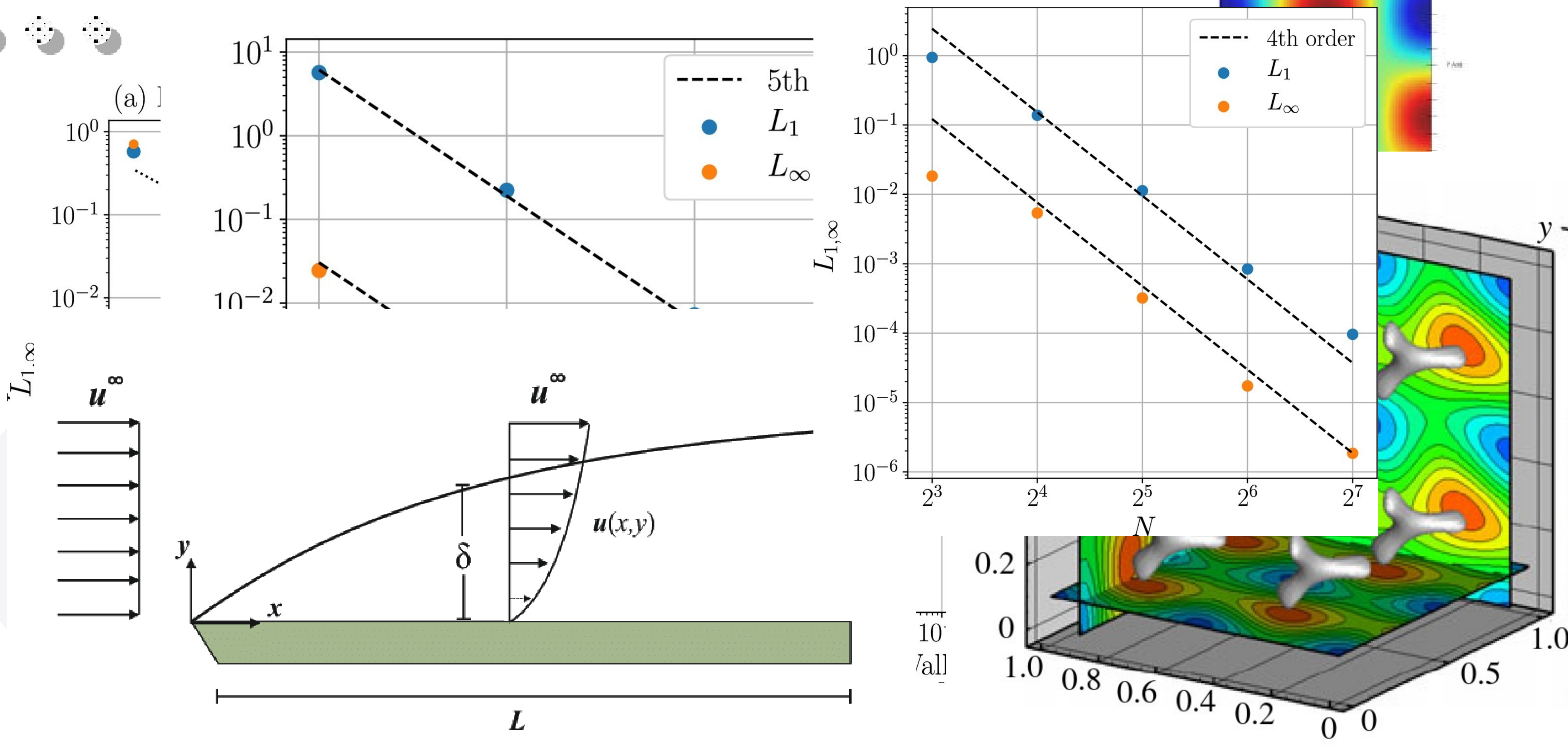




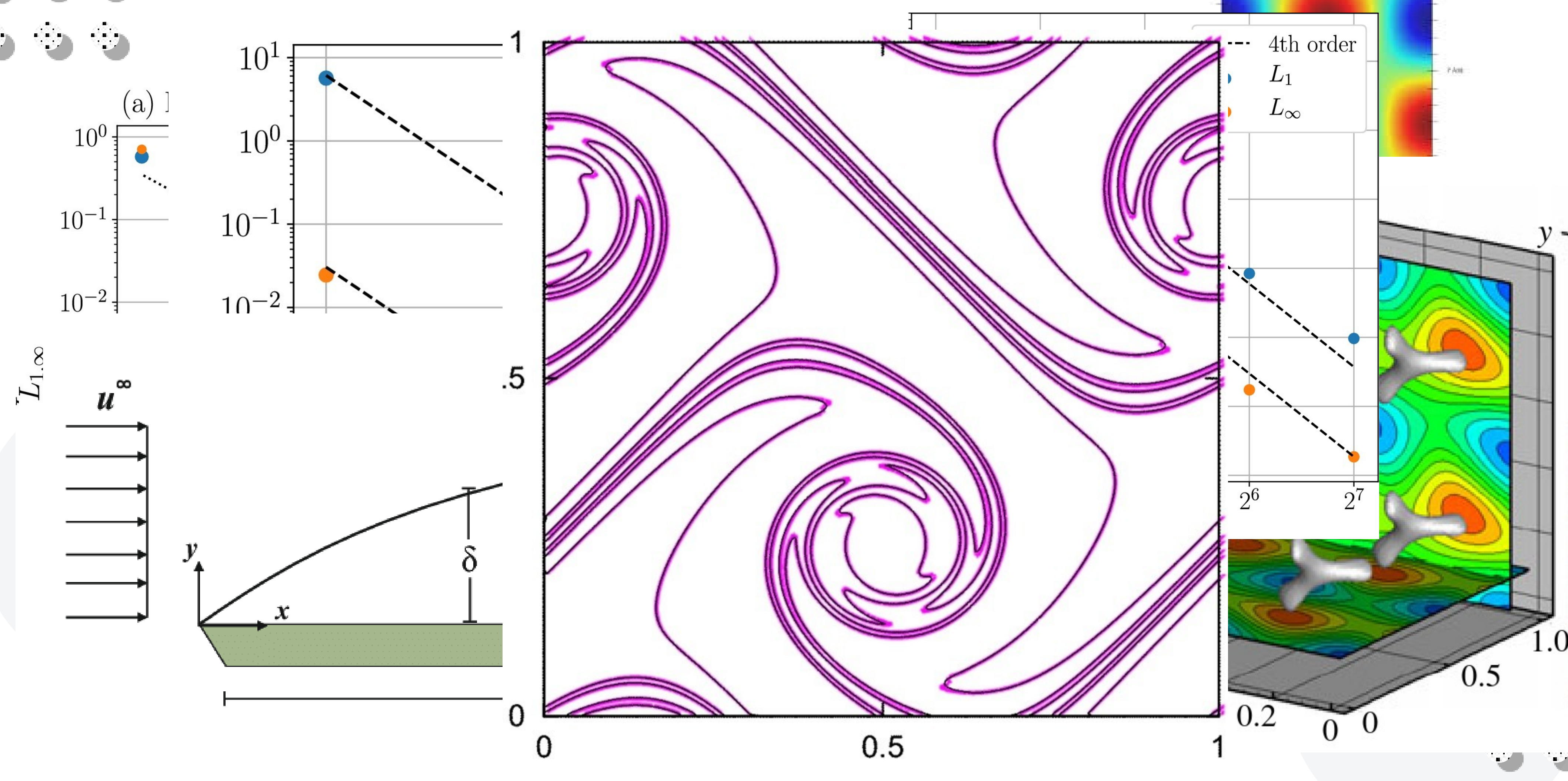
Test: 3) 3D vortex of Antuono (2020)



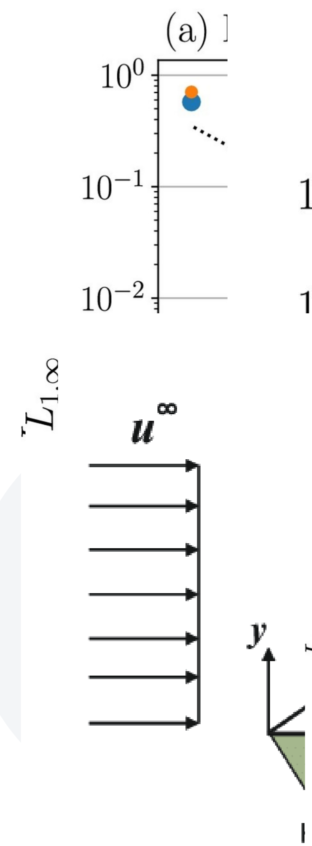
Test: 4) Viscous no-slip boundary layer

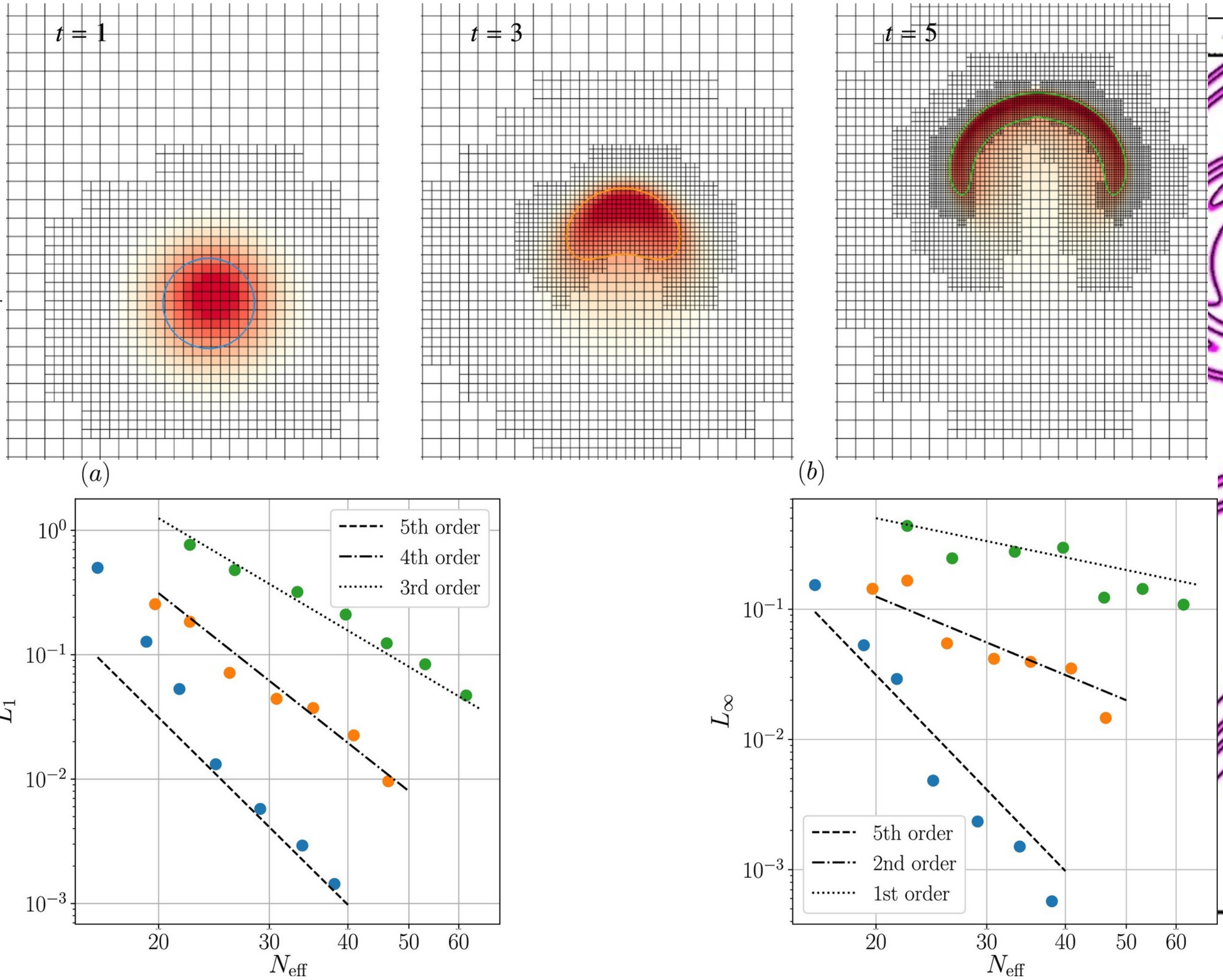


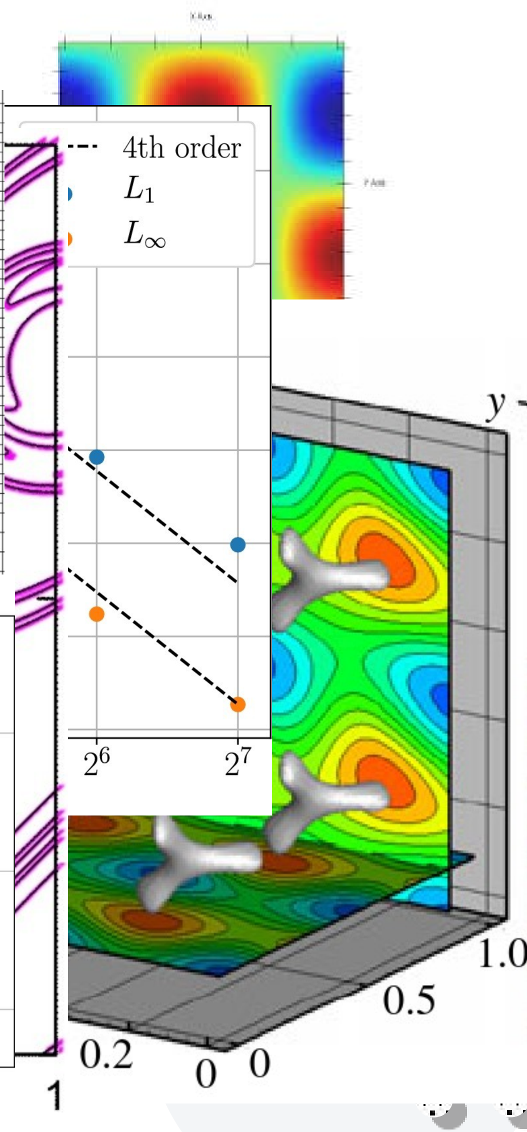
Test: 5) Unsteady vortex roll-up



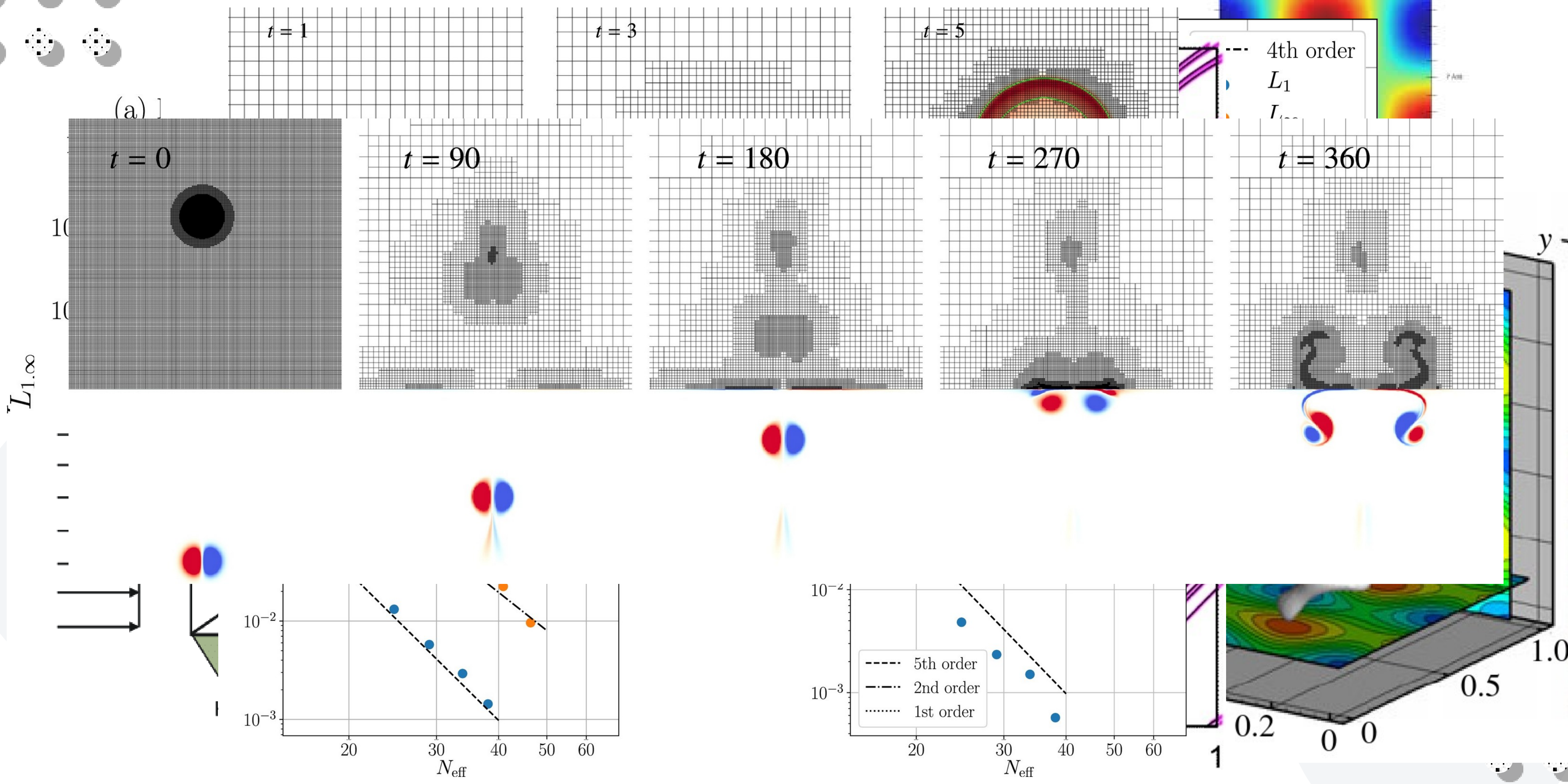
Test: 5) Rising buoyant plume



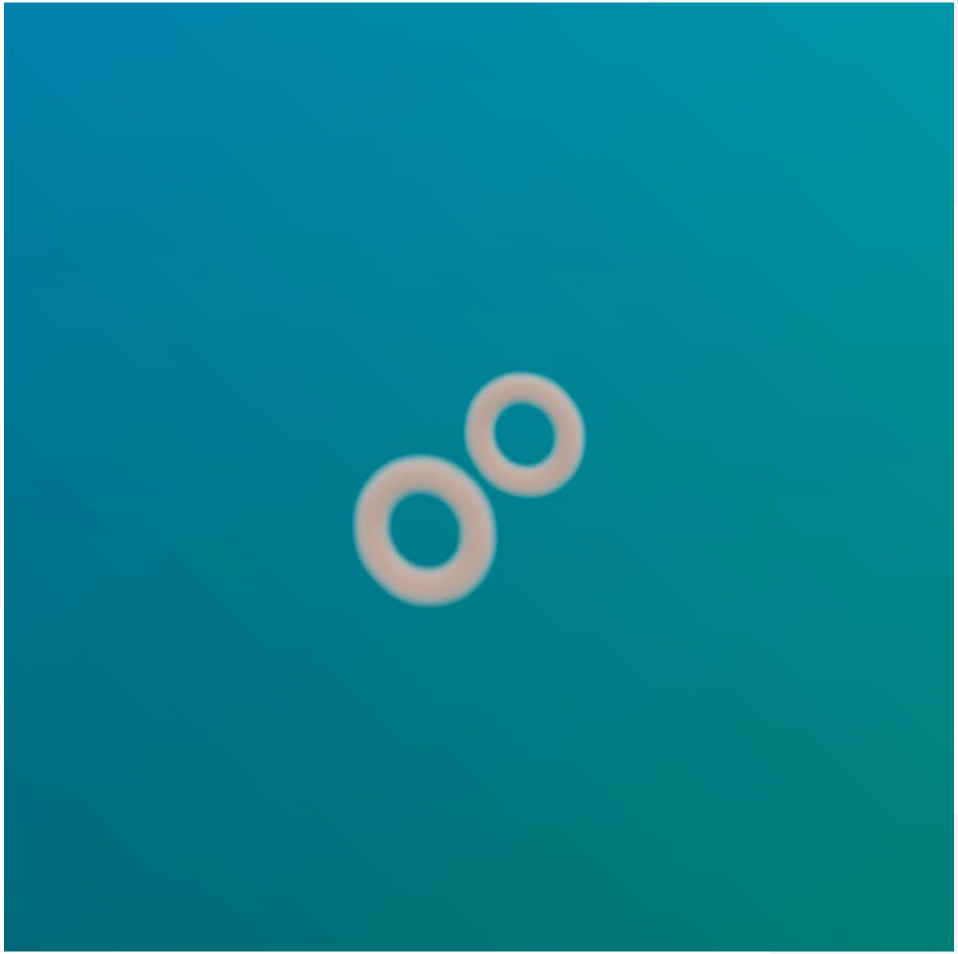




Test: 6) Vortex—Wall collision



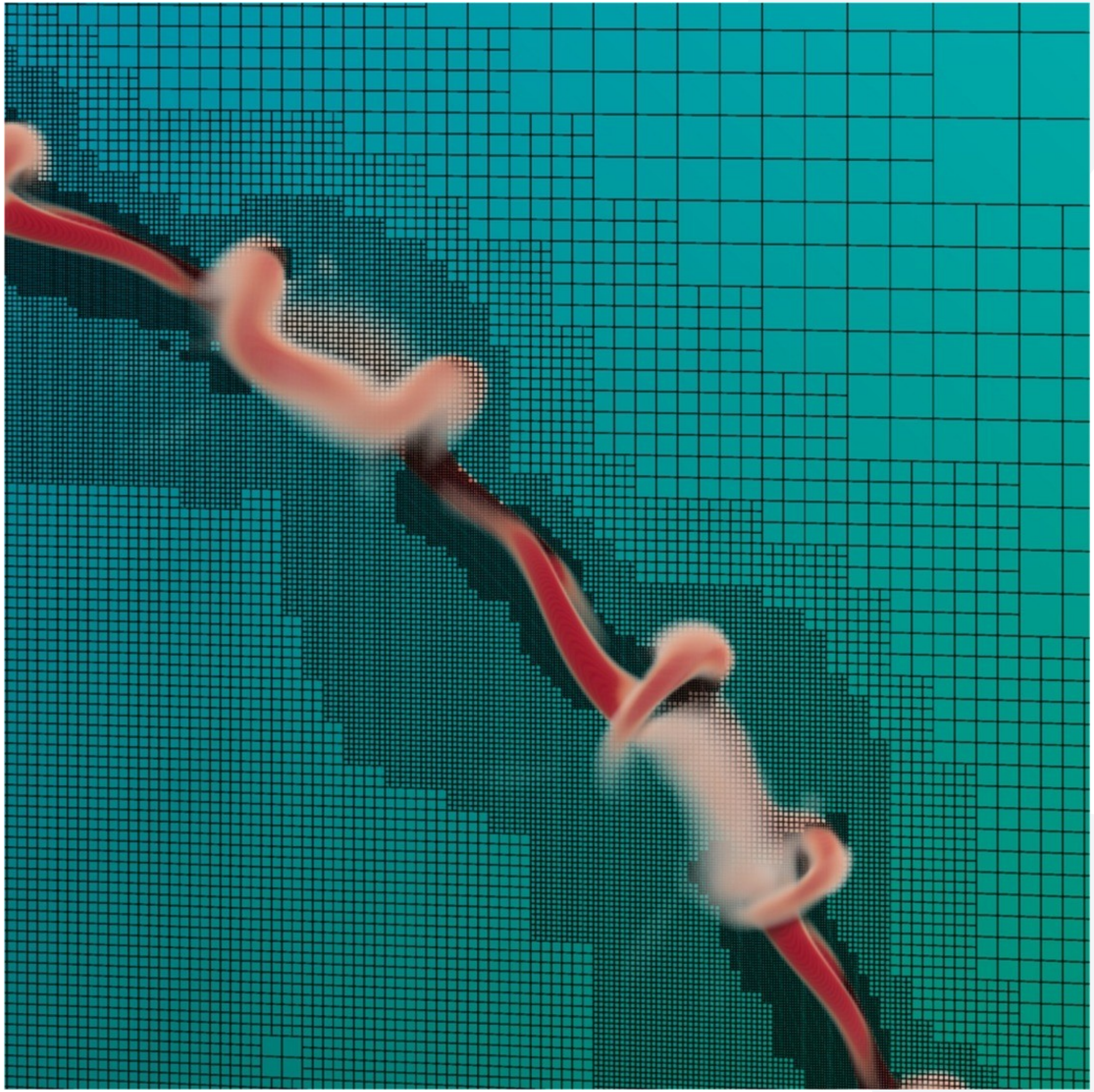
A comparison against `centered.h`



Movie:

www.basilisk.fr/sandbox/antoonvh/ring4.c

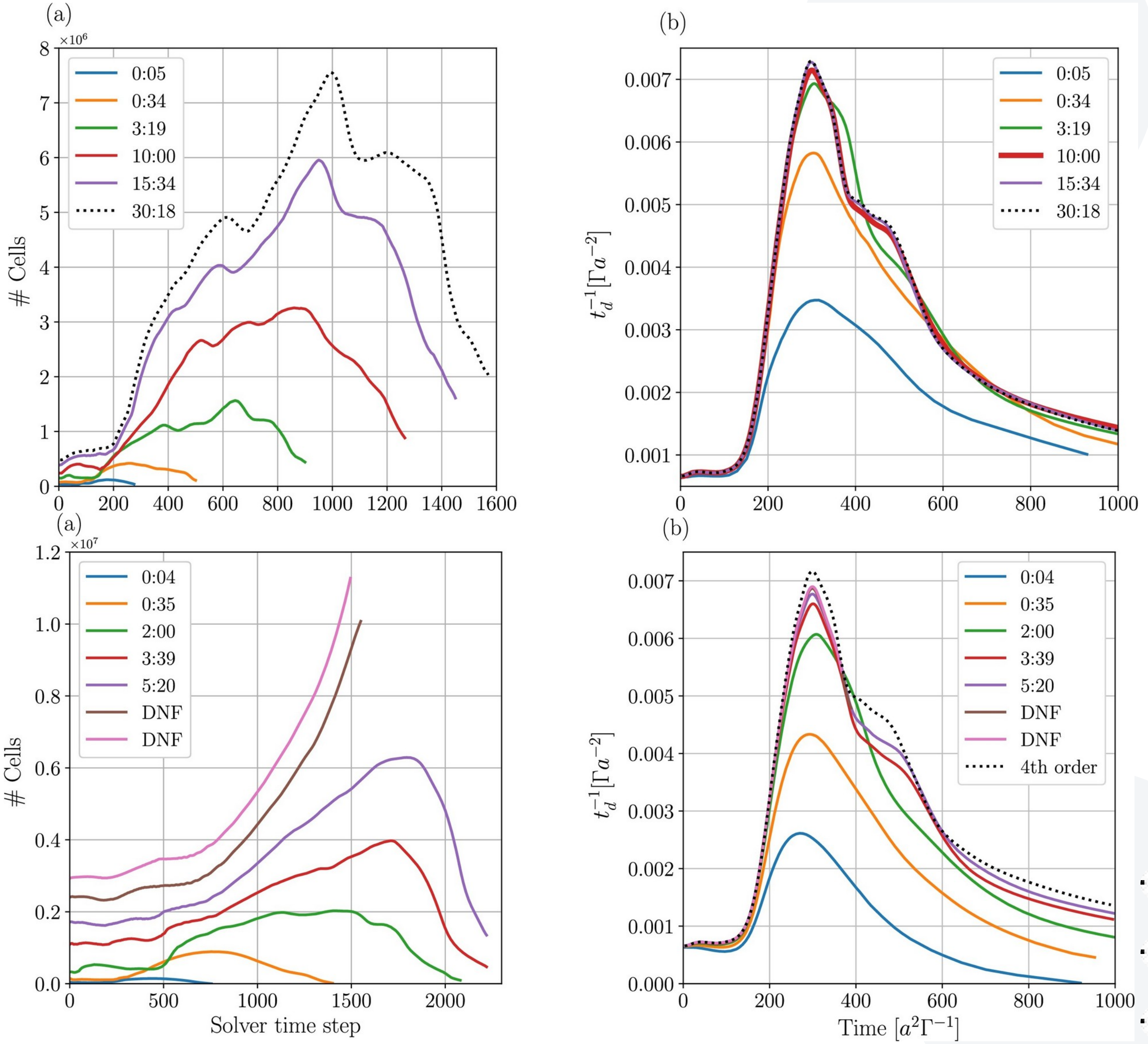
Refinement (k = 1)



How to converge on a tree grid?

sandbox/
antoonvh/
nsf4t.h

src/
navier-stokes/
centered.h



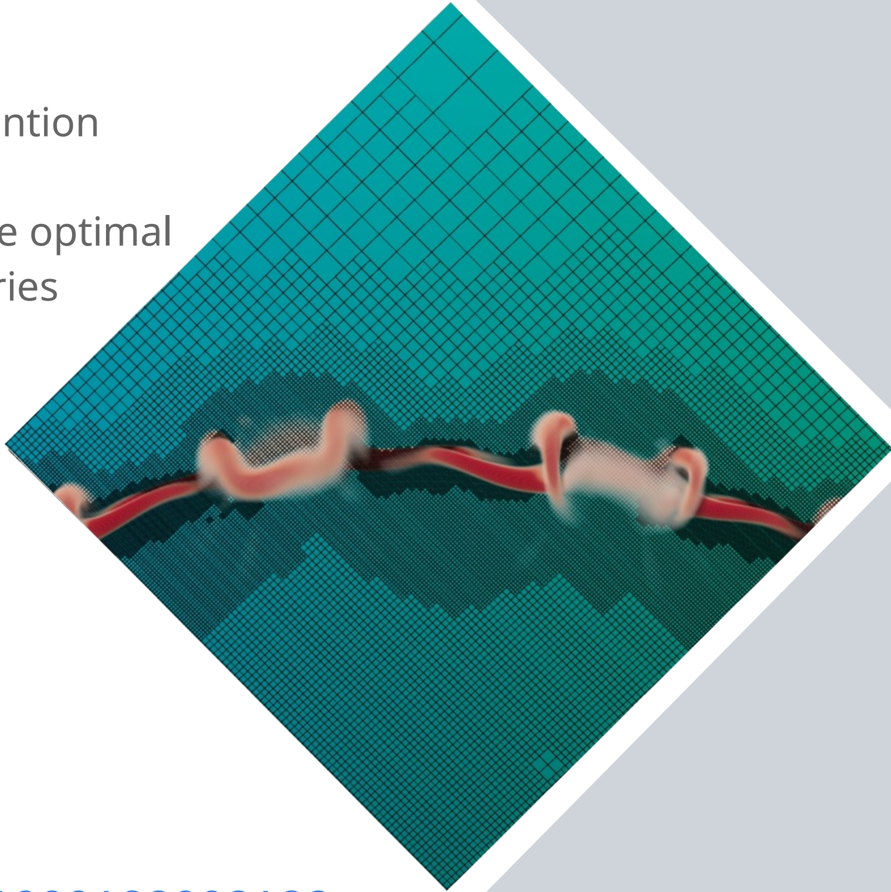
To summarize,

1. Treatment of resolution boundaries warrants special attention

2. Wavelet-Based refinement is powerful, but not likely to be optimal when directly linked to the treatment of resolution boundaries

3. A 4th-order accurate adaptive Navier-Stokes equations solver exists for Basilisk

4. Higher-order methods can be worth the hassle



van Hooft, J. A., & Popinet, S. (2022).

A fourth-order accurate adaptive solver for incompressible flow problems.

Journal of Computational Physics, 462, 111251.

https://www.sciencedirect.com/science/article/pii/S0021999122003138