A metric-based mesh adaptation method for elliptic equations based on quad/octree grids

L. Prouvost*, A. Belme, D. Fuster

Institut ∂'Alembert

BGUM 2023, 5–7 July, Paris

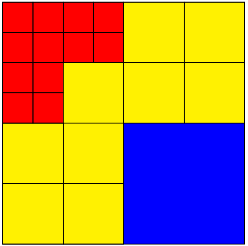
Motivation

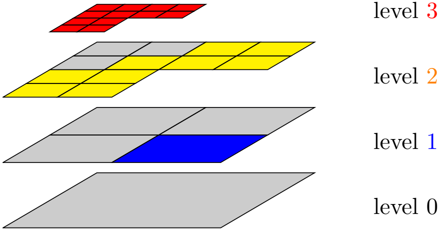
Optimize the grid distribution in Basilisk*

Extend and **improve** Basilisk adaptation method, including:
1: metric-based interpolation error
2: error introduced by numerical solver

Results obtained during my PhD

Quad/octree data structure



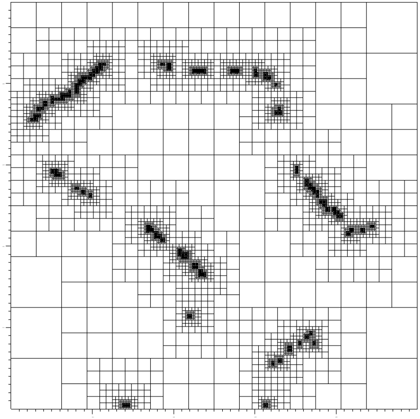


Grid size: $h = \frac{L_0}{2^{level}}$

*S. Popinet, 2015, http://basilisk.fr/

Adaptive Mesh Refinement (**AMR**)

The **mesh** plays a **crucial** role to limit numerical **errors**



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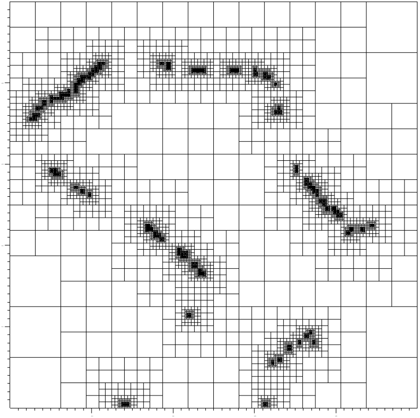

Compression ratio: $\eta \equiv \left(\frac{h_{min}}{\overline{h}}\right)^n$

Fixing h_{min} or η is equivalent for given N

Numerical errors: what are they ?

$$||u - \tilde{u}||_{L^p}$$

Continuous solution
Continuous PDE $\mathcal{L}u = s$

Discrete solution
Discrete PDE $\tilde{\mathcal{L}}\tilde{u} = \tilde{s}$

Numerical errors: what are they ?

$$||u - \tilde{u}||_{L^p} \leq ||u - \Pi_h u||_{L^p} + ||u'||_{L^p}$$

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Interpolate of u

$u' \equiv \tilde{u} - \Pi_h u$

Numerical errors: what are they ?

$$||u - \tilde{u}||_{L^p} \leq ||u - \Pi_h u||_{L^p} + ||u'||_{L^p}$$

$||u - \tilde{u}||_{L^p}$: u — Continuous solution, Continuous PDE $\mathcal{L}u = s$; $\tilde{u}$ — Discrete solution, Discrete PDE $\tilde{\mathcal{L}}\tilde{u} = \tilde{s}$

$\Pi_h u$ — Interpolate of u

$u' \equiv \tilde{u} - \Pi_h u$

Numerical error $\leq$ Interpolation error $+$ Implicit error

Interpolation error:
Local
Easy to estimate (Hessian of u)
Depend on u

Implicit error:
May propagate
More complex
Depend on u, PDE, discretization

Numerical errors

$$||u - \tilde{u}||_{L_p} \leq ||u - \Pi_h u||_{L_p} + ||u'||_{L_p}$$

with $u' \equiv \tilde{u} - \Pi_h u$

Numerical error < **Interpolation error**(u) + $\boldsymbol{u'}$(u,PDE,discretization)
(Implicit error)

Numerical errors

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Numerical error $<$ **Interpolation error**(u) + $\boldsymbol{u'}$(u,PDE,discretization)
(Implicit error)

Continuous PDE equation: $\mathcal{L}u = s$

Discretized PDE equation: $\tilde{\mathcal{L}}\tilde{u} = \tilde{s}$

By definition, u' depends on the PDE solved and its discretization

$$\tilde{\mathcal{L}}u' = s'$$

$s' = \tilde{s} - \tilde{\mathcal{L}}(\Pi_h u)$ represents a (local) source of error

Numerical errors

$$||u - \tilde{u}||_{L_p} \leq ||u - \Pi_h u||_{L_p} + ||u'||_{L_p}$$

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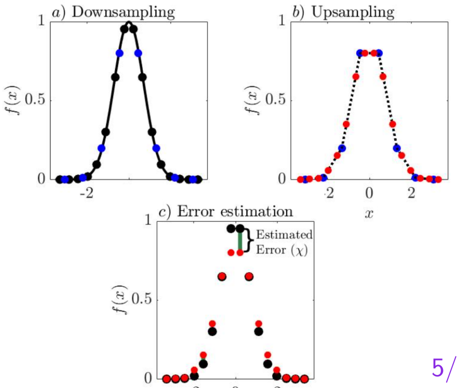
In AMR, we often consider $||u'||_{L^p} << ||u - \Pi_h u||_{L^p}$

and search the mesh minimizing $||u - \Pi_h u||_{L^p}$ to minimize $||u - \tilde{u}||_{L^p}$

Interpolation errors

Wavelet-based[1] $\rightarrow$ $\boldsymbol{L}^{\infty}$-norm of the error

But for numerical solutions, $\boldsymbol{L}^2$-norm error is generally recommended[2]



[1] J. A. van Hooft et al., 2018
[2] F. Alauzet and A. Loseille, 2016

Interpolation errors

Metric-based $\rightarrow$ L^p-norm error

Wavelet-based[1] $\rightarrow$ L^∞-norm of the error

Particularization of [2] to quadtree/octree grids [3]

$$|u - \Pi_h u|(\mathbf{x}) = A_{loc}\left(tr(|H|(\mathbf{x}))\right)^{\frac{n}{2p+n}} \overline{h}^2$$

$$||u - \Pi_h u||_{\Omega, L^p} = A_{global}\overline{h}^2$$

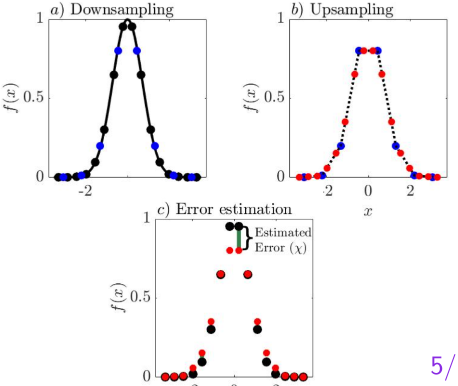
But for numerical solutions, L^2-norm error is generally recommended[2]

A_{loc} and A_{global} depend on:

p: Error norm

n: Problem dimension

H: Hessian of u



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Interpolation errors

Metric-based $\rightarrow$ $\boldsymbol{L^p}$-norm error

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$$|u - \Pi_h u|(\boldsymbol{x}) = C_n \left(\int_\Omega \left(tr(|H|(\boldsymbol{x}))\right)^{\frac{np}{2p+n}} \, \mathrm{d}\boldsymbol{x} \right)^{\frac{2}{n}} \left(tr(|H|(\boldsymbol{x}))\right)^{\frac{n}{2p+n}} N^{-\frac{2}{n}}$$

$$||u - \Pi_h u||_{\Omega, L^p} = C_n \left(\int_\Omega \left(tr(|H|(\boldsymbol{x}))\right)^{\frac{np}{2p+n}} \, \mathrm{d}\boldsymbol{x} \right)^{\frac{2p+n}{np}} N^{-\frac{2}{n}}$$

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Local

Quantify the min error for given N

Do not depend on the equation

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Used as reference to evaluate AMR performances

p: Error norm

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Numerical errors

Numerical error = Interpolation error??

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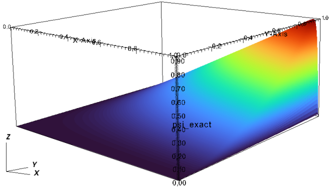
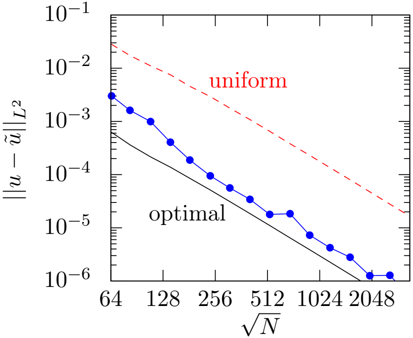
Sometimes YES

Example: solution Helmholtz-Poisson equation

$$\nabla \cdot (D \nabla u) - u = s(x)$$

Boundary layer problem





Numerical errors

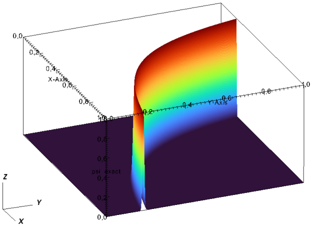
Numerical error = Interpolation error??

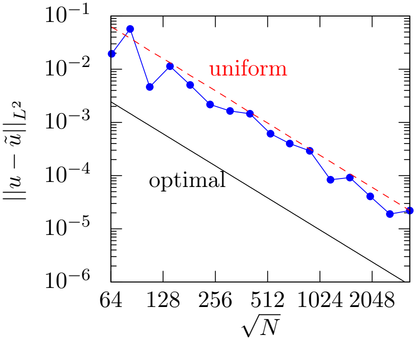
Sometimes NOT

Example: solution Helmholtz-Poisson equation

$$\boxed{\nabla \cdot (D\nabla u) - u = s(x)}$$

$$u_A = \exp\left(-\left(\frac{xy-a}{\kappa^2}\right)^2\right)$$





Numerical errors

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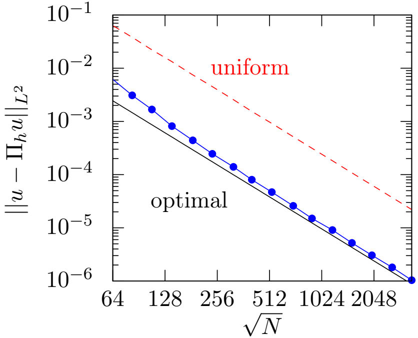
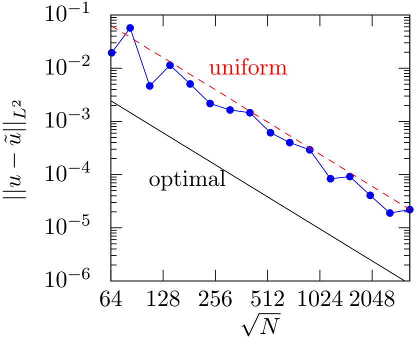
Example: solution Helmholtz-Poisson equation



Interpolation errors only (injecting u on nodes)



Total numerical error

Numerical errors

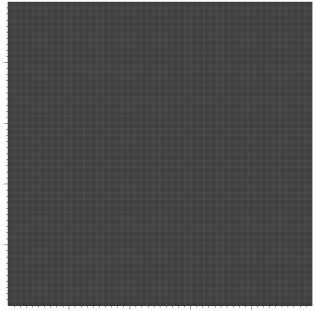
Numerical experiment: Start with a uniform grid ($\eta = 1$)

We minimize the interpolation error fixing h_{min}!

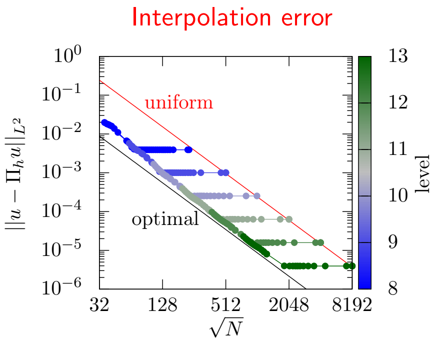
We change η until getting the optimal grid ($\eta = \eta_0$)

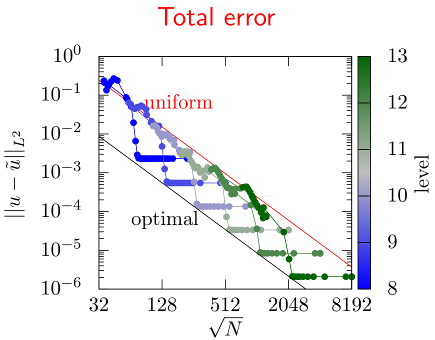
Computation of: Interpolation error (injecting the exact solution)

Exact error (solving PDE)



Numerical errors





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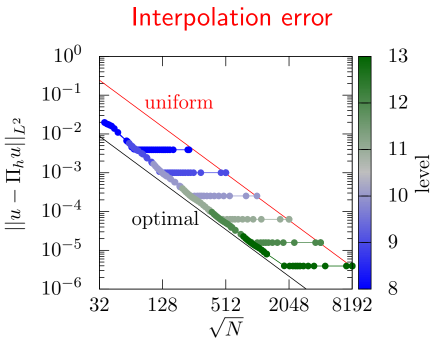

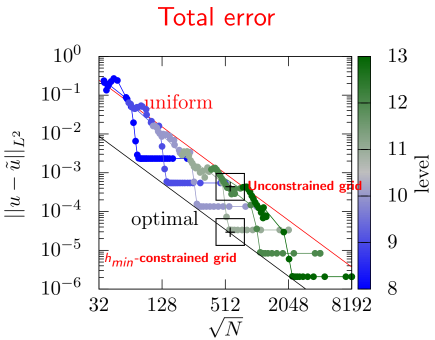

Numerical errors

Numerical error $\leq$ Interpolation error + u'

$$\tilde{\mathcal{L}} u' = s'$$

How s' and u' behaves for Helmholtz-Poisson equation??

$$D\tilde{\nabla}^2 \tilde{u}'_i + \lambda \tilde{u}'_i = s_i - D(\tilde{\nabla}^2(\Pi_h u))_i - \lambda u_i$$

$$\boxed{s' = D\left[(\nabla^2 u)_i - \tilde{\nabla}^2(\Pi_h u)_i\right]}$$

In 1D for non uniform mesh: $s'_i = \frac{1}{2} D \frac{\partial^2 u}{\partial x^2}\left(1 - \frac{h_{i+1}+h_{i-1}}{2h_i}\right) + \mathcal{O}(h_i^2)$

Hessian of the solution

Weighted by element size variation

Numerical errors

Numerical error $\leq$ **Interpolation error** $+$ $\boldsymbol{u'}$

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Hessian of the solution

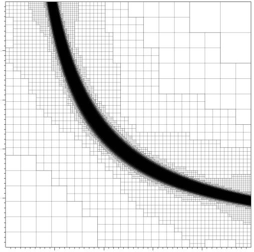
Weighted by element size variation

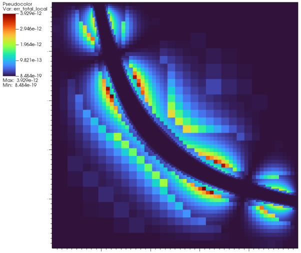
A **mesh minimizing** $||u - \Pi_h u||_{L^p}$ **may** have cell sizes varying too fast and **produce high** $||u'||_{L^p}$

Numerical errors

Unconstrained grid

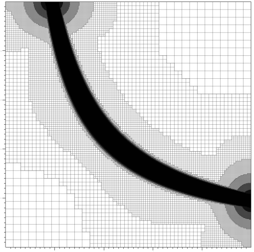
Total error map

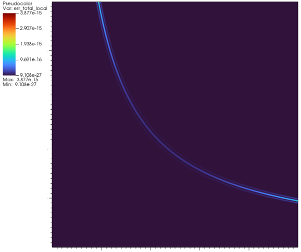




h_{min}-Constrained grid (same N)

Int err +1%, Tot. Err -95%



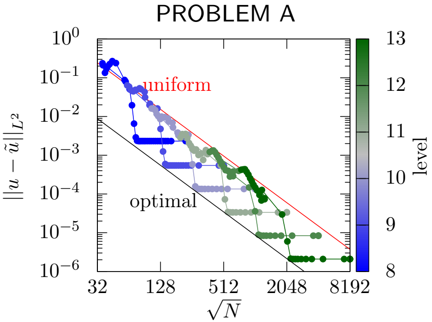


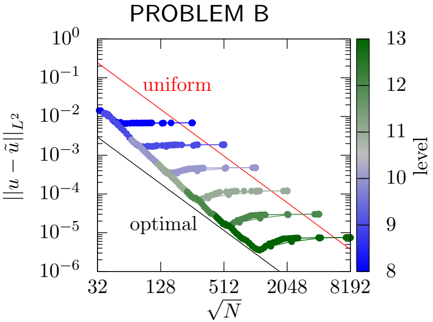
Numerical errors

Alternative (N independent) representation of the total error

y-axis: $\gamma_2 \equiv \frac{\text{Total error}}{\text{Optimal Interp error with same N}}$ (Normalized error: AMR grid performance)

x-axis: $\eta \equiv \left(\frac{h_{min}}{\bar{h}}\right)^n$ compression ratio





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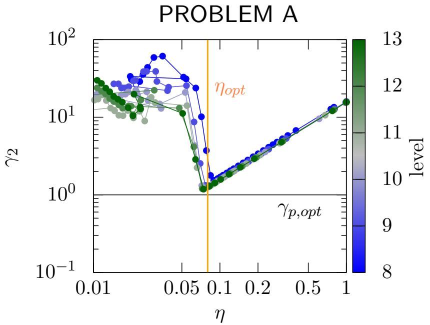
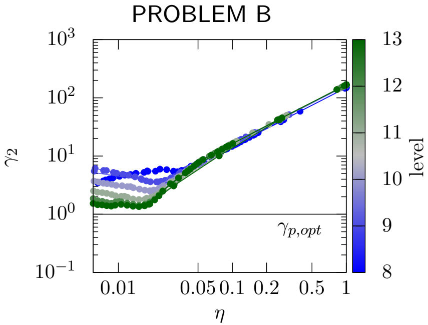
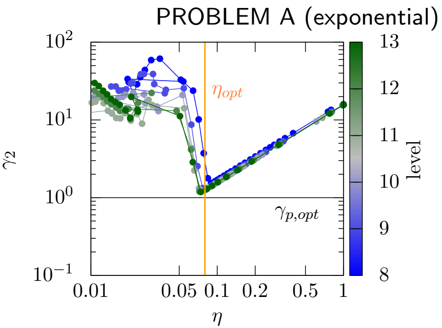


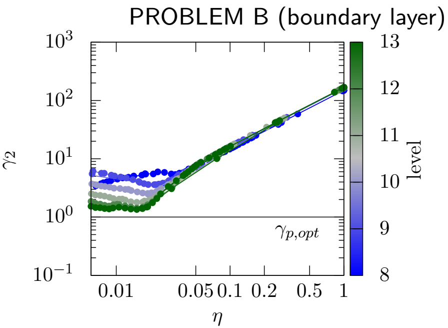


$\eta = 1$ Uniform grid

Numerical errors

Alternative (N independent) representation of the total error





$\eta < \eta_{opt}$

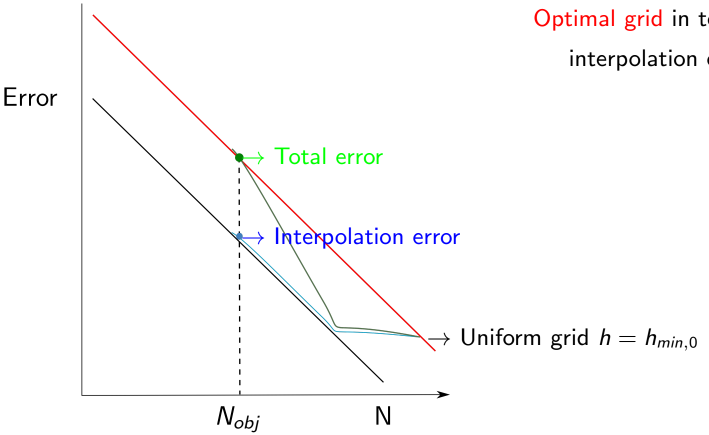
Interp. error does not work

$\eta \geq \eta_{opt}$

Interp. error works

Numerical errors

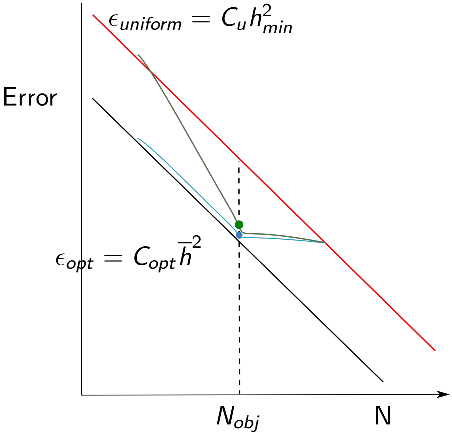
UNRESTRICTED GRID



Optimal grid in terms of interpolation error

Numerical errors

Solution: Restrict η (or equivalently h_{min})



Numerical errors

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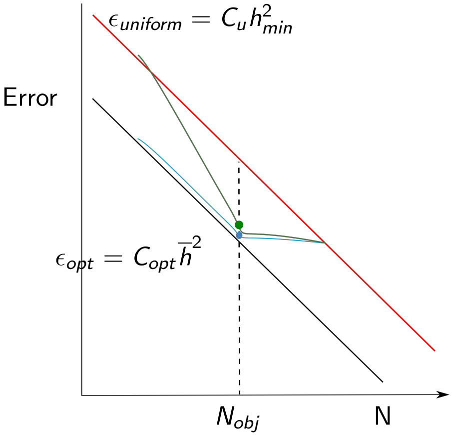
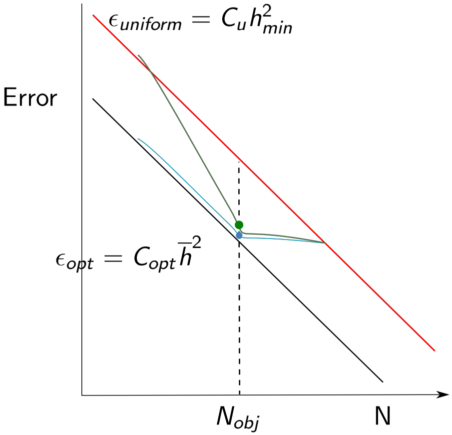


$\epsilon_{uniform} = \epsilon_{opt}$

Numerical errors

Solution: Restrict η (or equivalently h_{min})

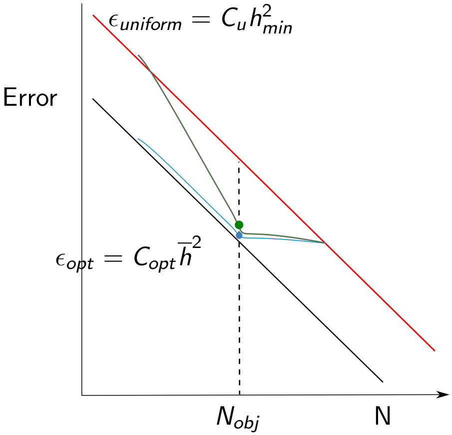


$$\epsilon_{uniform} = \epsilon_{opt}$$

$$\eta_c = \frac{\left(\int_\Omega (\mathrm{tr}(|H|(\boldsymbol{x})))^{\frac{np}{2p+n}} \, \mathrm{d}\boldsymbol{x} \right)^{\frac{2p+n}{2p}}}{L_0^n \left(\int_\Omega \mathrm{tr}(|H|(\boldsymbol{x}))^p \, \mathrm{d}\boldsymbol{x} \right)^{\frac{n}{2p}}}$$

Numerical errors

Solution: Restrict η (or equivalently h_{min})



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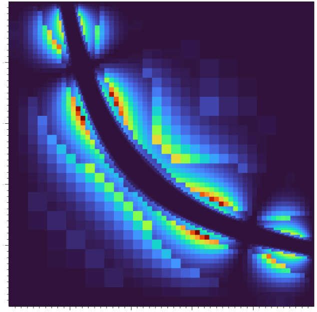
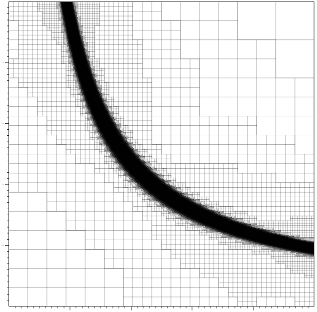
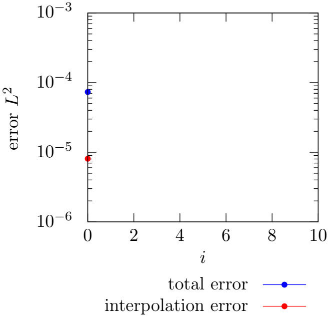
$$h_{min} = L_0 \left(\frac{\eta_c}{N_{obj}} \right)^{\frac{1}{n}}$$

Reduce numerical errors

Solution: Restrict η (or equivalently h_{min})







Reduce numerical errors

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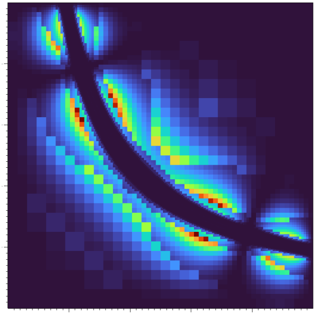
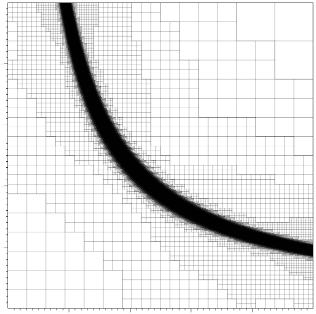
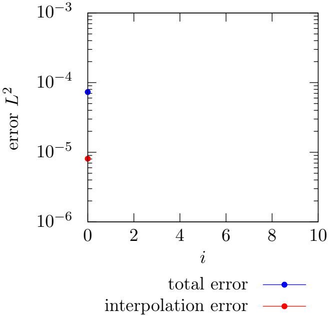


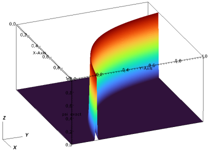


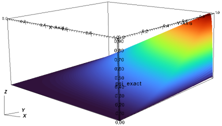


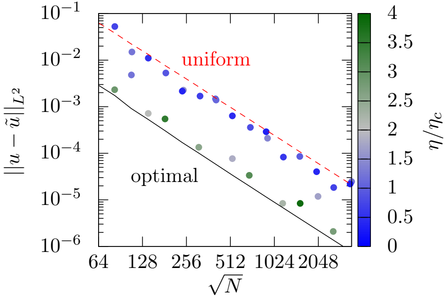
Final mesh: not optimal in interpolation error but reduce total numerical error

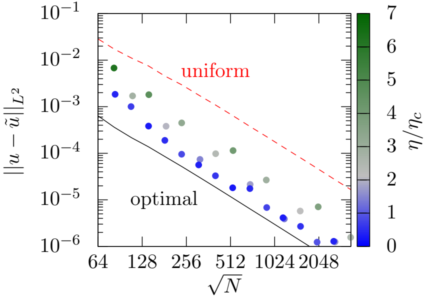
Numerical errors

Grid convergence under $h_{min} - \eta$ restriction









How to use it ?

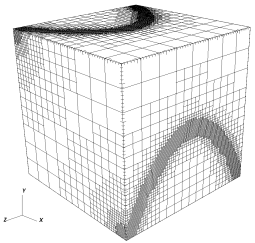
```
/* restriction on the minimum grid size (optional) */
double etaopt = estimate_eta_opt(2, {psi});
maxlevel = 0.5*log(Nobj/etaopt)/log(2.);   // maxlevel: global variable. Optional.

/* epsilon criteria for cell refinement/coarsening (mandatory) */
AMReps = 0.01;                              // AMReps: global variable

/* AMR */
adapt_metric( {psi} );                      // user interface similar to adapt_wavelet()
```

More details in my Basilisk sandbox

http://basilisk.fr/sandbox/prouvost/README

Conclusions

- Local AMR method for quadtree/octree grids.
- A constrain on h_{min} is compulsory to guarantee good performance.
- h_{min} can be theoretically estimated
- easy-to-use for the user-interface



Perspectives

- Test other equations
- Minimize propagation errors too?

A metric-based mesh adaptation method for elliptic equations based on quad/octree grids

L. Prouvost*, A. Belme, D. Fuster

Institut ∂'Alembert

BGUM 2023, 5–7 July, Paris