Falling liquid film control

via linear quadratic regulation

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Basilisk/Gerris User's Meeting, 6th July 2023

Introduction
Motivation

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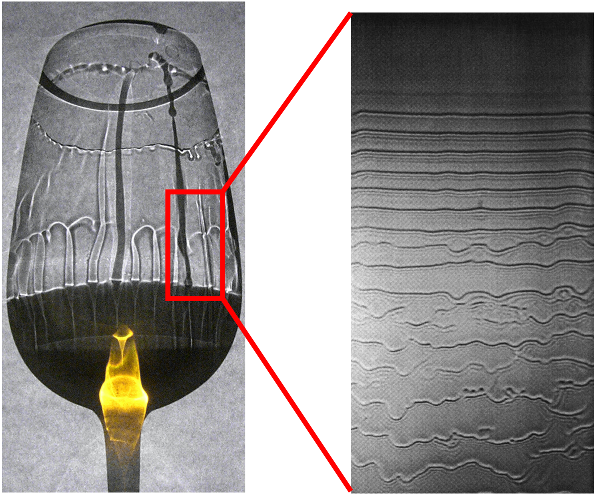
Motivation



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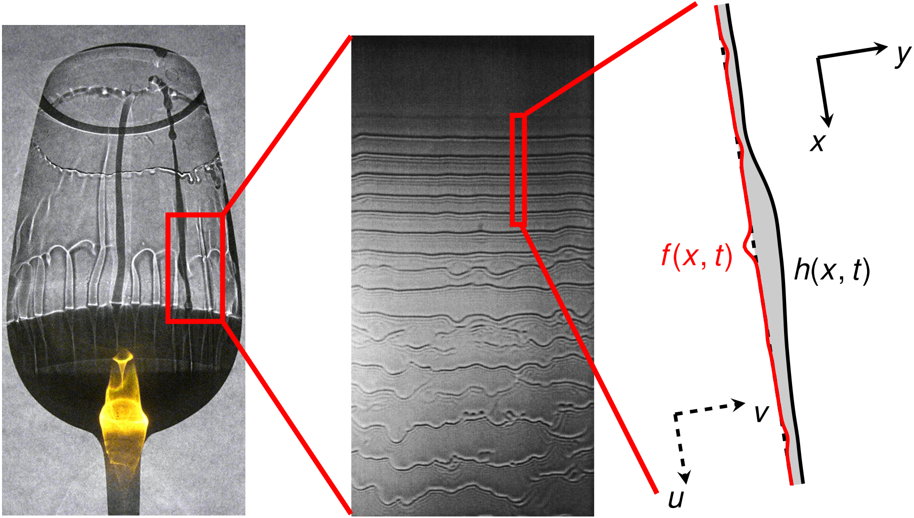
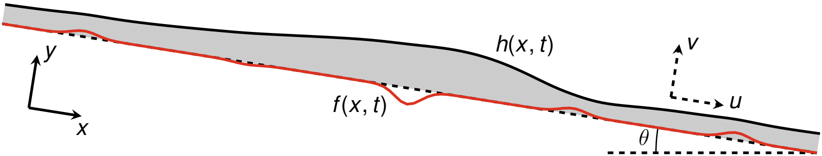


Introduction

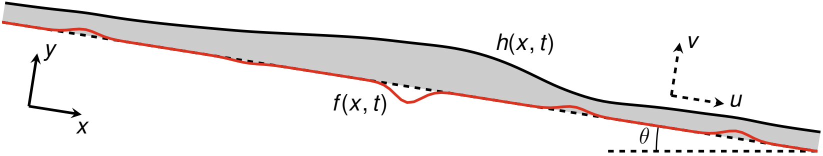
Navier-Stokes film



Thin fluid film falling down an inclined plane

Introduction

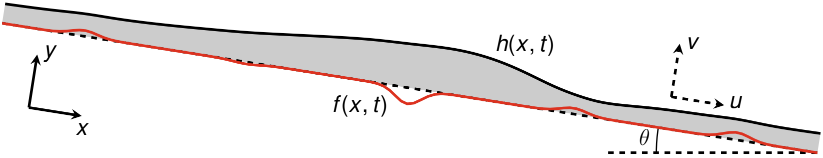
Navier-Stokes film



Thin fluid film falling down an inclined plane

Our aim is to stabilise the uniform film solution by injecting and removing fluid from the base at a finite number of actuators.

Introduction
Navier-Stokes film

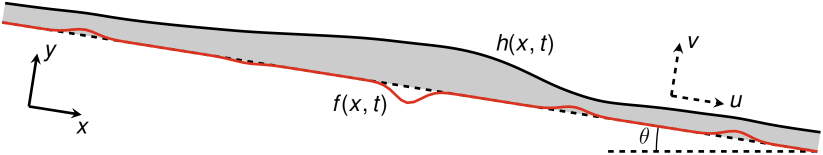


Thin fluid film falling down an inclined plane

Navier-Stokes flow in the fluid

$$Re(u_t + uu_x + vu_y) = -p_x + 2 + u_{xx} + u_{yy},$$
$$Re(v_t + uv_x + vv_y) = -p_y - 2\cot\theta + v_{xx} + v_{yy},$$
$$u_x + v_y = 0.$$

Introduction
Navier-Stokes film



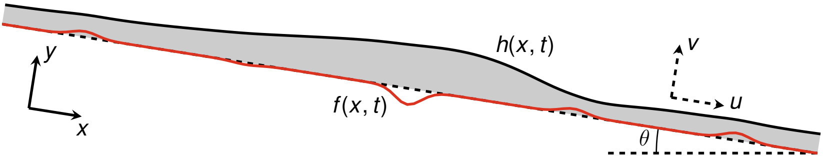
Thin fluid film falling down an inclined plane

Boundary conditions at the base

$$u = 0, \qquad v = f(x,t).$$

Introduction

Navier-Stokes film



Thin fluid film falling down an inclined plane

At the interface, $y = h(x,t)$, the nonlinear dynamic stress balance

$$(v_x + u_y)(1 - h_x^2) + 2h_x(v_y - u_x) = 0,$$

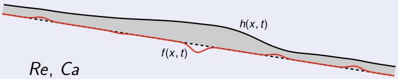
$$p - \frac{2}{1 + h_x^2}(v_y + u_x h_x^2 - h_x(v_x + u_y)) = -\frac{1}{Ca}\frac{h_{xx}}{(1 + h_x^2)^{3/2}},$$

and the kinematic boundary condition

$$h_t = v - uh_x.$$

Introduction

Navier-Stokes film



- multi-phase flow
- complex boundary conditions
- highly nonlinear
- computationally expensive

Introduction

Feedback control

The most **general feedback control** problem looks like

$$x_t = \mathcal{A}x + \mathcal{B}u, \qquad u = \mathcal{K}y, \qquad y = \mathcal{C}x.$$

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to give a system of linear ODEs.

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Introduction

Linear quadratic regulator control

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The choice of K is currently not unique, so we introduce a **quadratic cost**

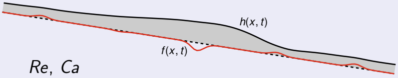
$$c = \int_0^\infty x^\mathsf{T} U x + u^\mathsf{T} V u \, \mathrm{d}t,$$

thus forming an LQR problem.

Introduction

Problem of control

Navier-Stokes film



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LQR controls

- cheap to compute
- easy to design

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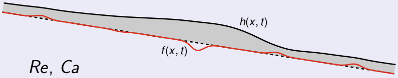

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But requires a linear system of ODEs

Hierarchical framework

Reduced order model

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$$\epsilon = \frac{|\nabla h|}{h} \ll 1,$$

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$$\begin{aligned} h_t + q_x &= f, \\ \frac{2Re}{5} h^2 q_t + q &= \frac{h^3}{3}\left(2 - 2h_x \cot\theta + \frac{h_{xxx}}{Ca}\right) \\ &\quad + Re\left(\frac{18q^2 h_x}{35} - \frac{34hqq_x}{35} + \frac{hqf}{5}\right). \end{aligned}$$

These are the weighted-residual integral boundary layer equations.

Hierarchical framework

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These are the WR equations.

Hierarchical framework

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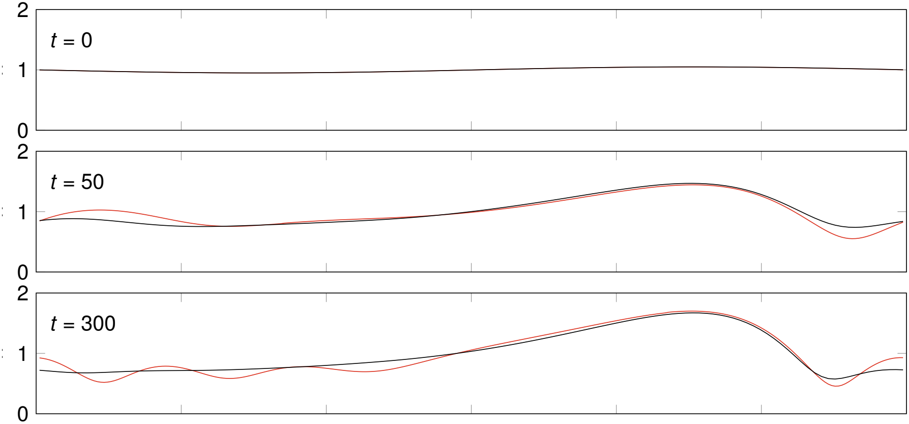

Development of travelling wave for **Navier-Stokes** and **WR** systems. $Re = 10$, $Ca = 0.05$.

Hierarchical framework

Linearisation

$$
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h_t + q_x &= f, \\
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These equations are still **very nonlinear**.

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These equations are still **very nonlinear**. Assuming that any perturbations from the uniform film are small

$$h = 1 + \delta\hat{h}, \qquad q = \frac{2}{3} + \delta\hat{q}, \qquad f = \delta\hat{f},$$

Hierarchical framework

Linearisation

$$h_t + q_x = f,$$

$$\frac{2Re}{5}h^2 q_t + q = \frac{h^3}{3}\left(2 - 2h_x \cot\theta + \frac{h_{xxx}}{Ca}\right) + Re\left(\frac{18q^2 h_x}{35} - \frac{34hqq_x}{35} + \frac{hqf}{5}\right).$$

These equations are still **very nonlinear**. Assuming that any perturbations from the uniform film are small

$$h = 1 + \delta\hat{h}, \qquad q = \frac{2}{3} + \delta\hat{q}, \qquad f = \delta\hat{f},$$

we have

$$\hat{h}_t = -\hat{q}_x + \hat{f},$$

$$\hat{q}_t = \left[\frac{5}{Re} + \left(\frac{4}{7} - \frac{5\cot\theta}{3Re}\right)\partial_x + \frac{5}{6ReCa}\partial_{xxx}\right]\hat{h} - \left[\frac{5}{2Re} + \frac{34}{21}\partial_x\right]\hat{q} + \frac{1}{3}\hat{f}.$$

Hierarchical framework

Discretisation

Finally we can discretise

$$\begin{bmatrix} h \\ q \end{bmatrix}_t = \begin{bmatrix} J_{hh} & J_{hq} \\ J_{qh} & J_{qq} \end{bmatrix} \begin{bmatrix} h \\ q \end{bmatrix} + \begin{bmatrix} \Psi_h \\ \Psi_q \end{bmatrix} f,$$

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$$= \left(\begin{bmatrix} J_{hh} & J_{hq} \\ J_{qh} & J_{qq} \end{bmatrix} + \begin{bmatrix} \Psi_h \\ \Psi_q \end{bmatrix} K \right) \begin{bmatrix} h \\ q \end{bmatrix}.$$

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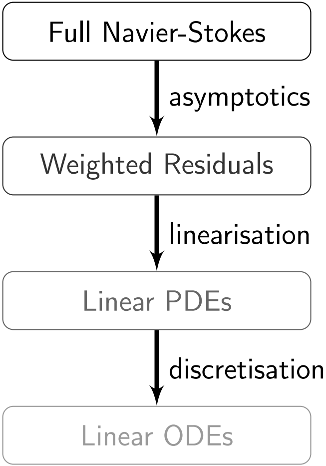
Or, in more concise notation,

$$\xi_t = (J + \Psi K)\,\xi$$

Hierarchical framework



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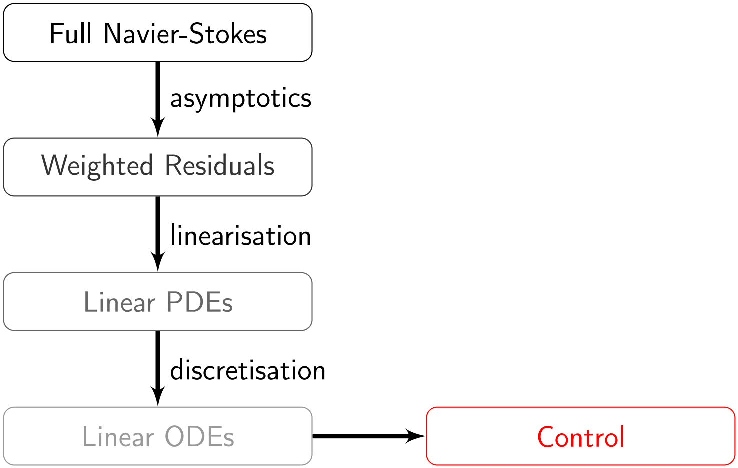

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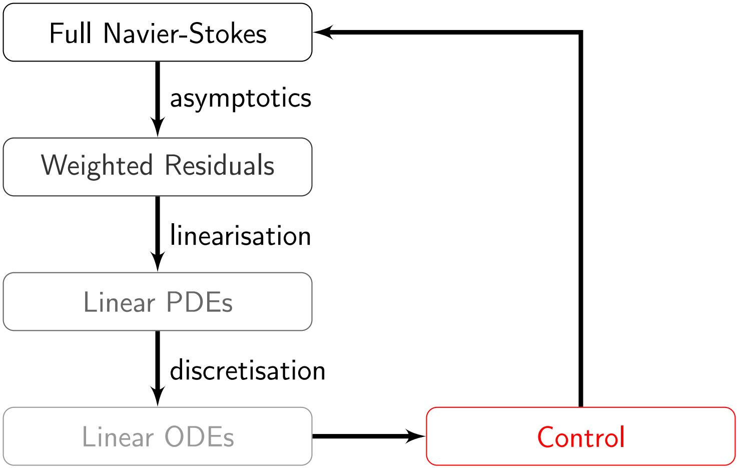
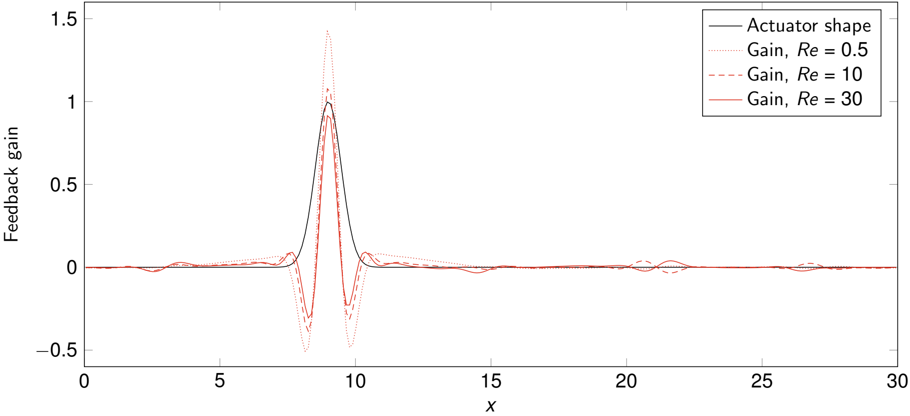


Numerical experiments

What does this actually look like in practice?

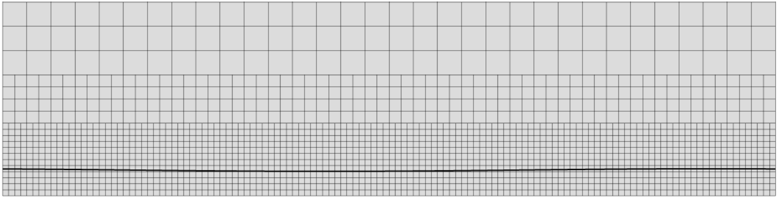
Numerical experiments

Gain matrix



Feedback gain for a single **actuator**. Re various, $Ca = 0.05$.

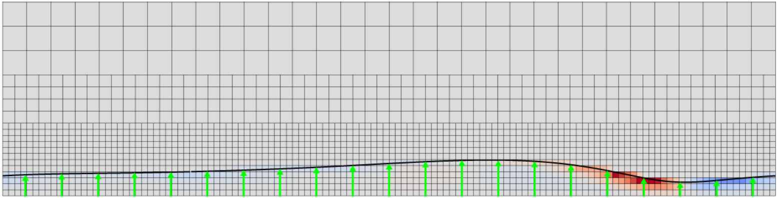
Numerical experiments
Simulation



Initial development of a travelling wave. $Re = 15,\ Ca = 0.05.$

Numerical experiments

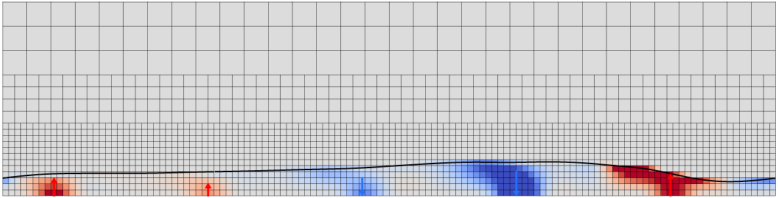
Simulation



Measurement of the **height**. $Re = 15,\ Ca = 0.05.$

Numerical experiments

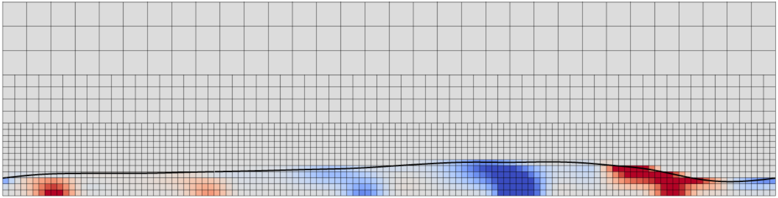
Simulation



Computation of **controls**. $Re = 15,\ Ca = 0.05.$

Numerical experiments

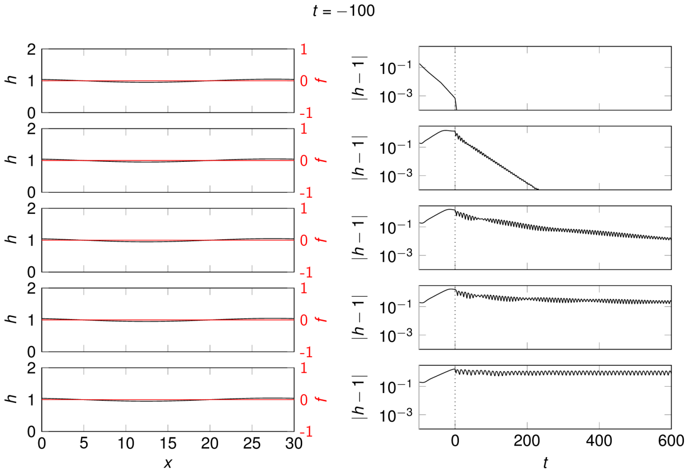
Simulation



Controls stabilising the uniform film. $Re = 15,\ Ca = 0.05$.

Numerical experiments

Successful control



Controls attempting to stabilise the uniform film. *Re* various, $Ca = 0.05$.

Numerical experiments

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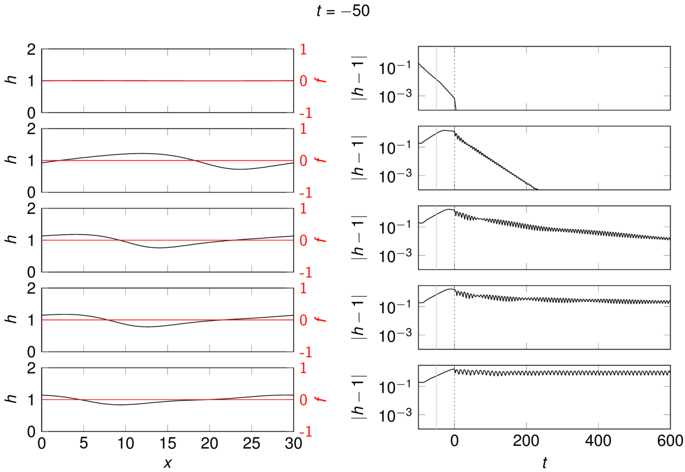

Controls attempting to stabilise the uniform film. *Re* various, $Ca = 0.05$.

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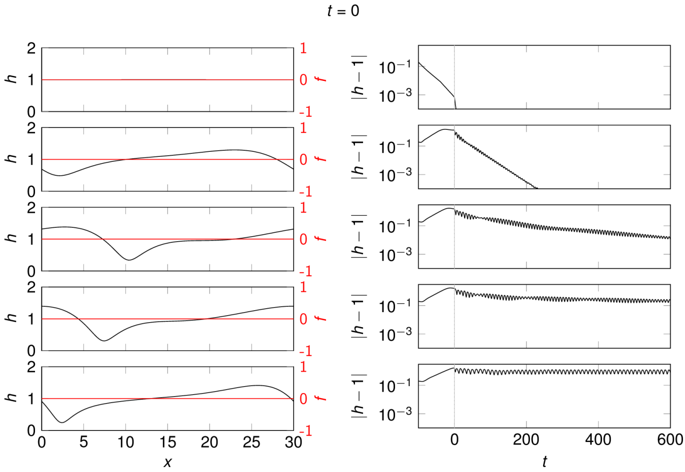

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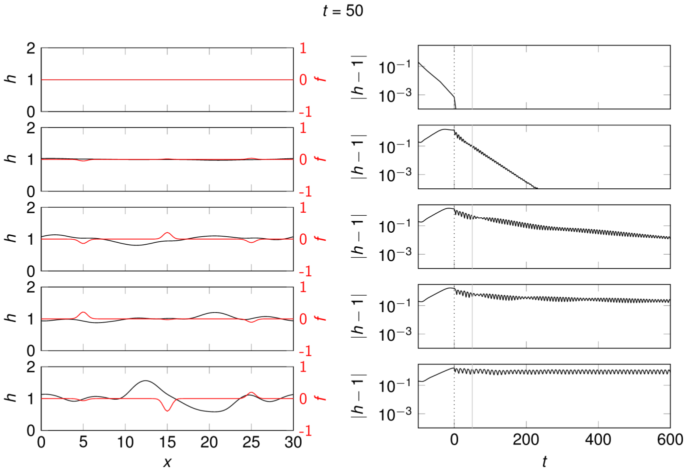

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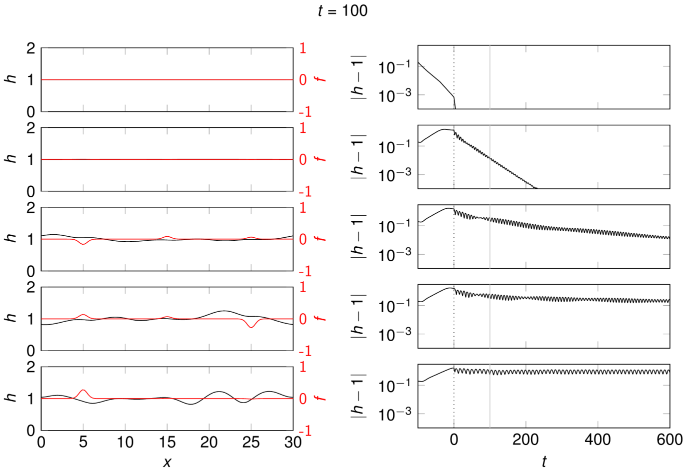

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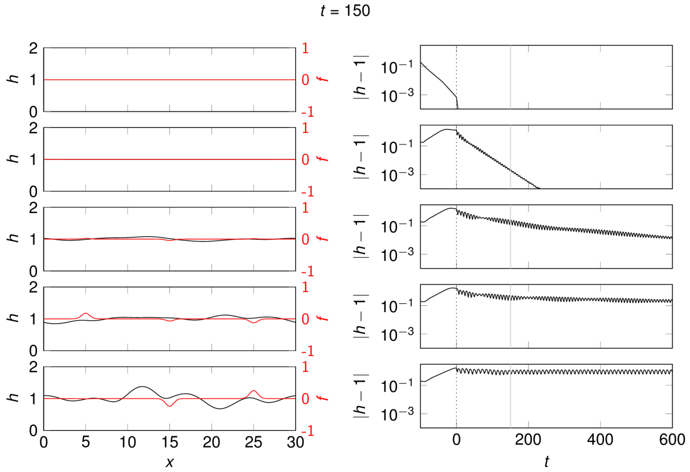

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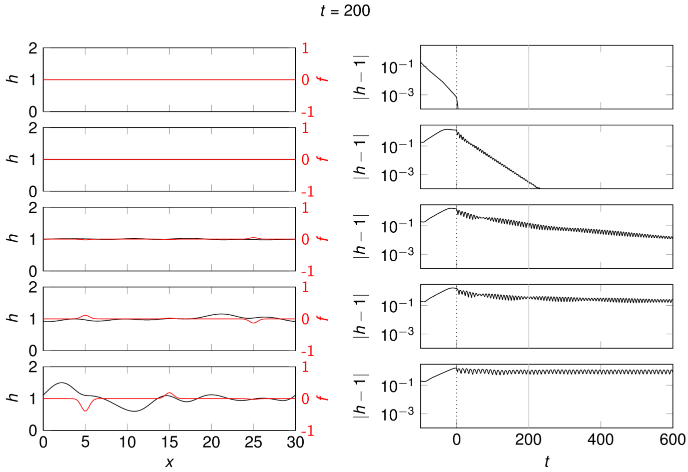

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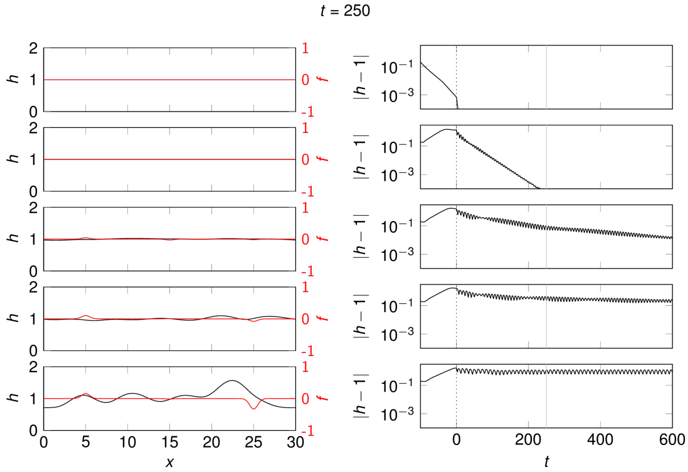

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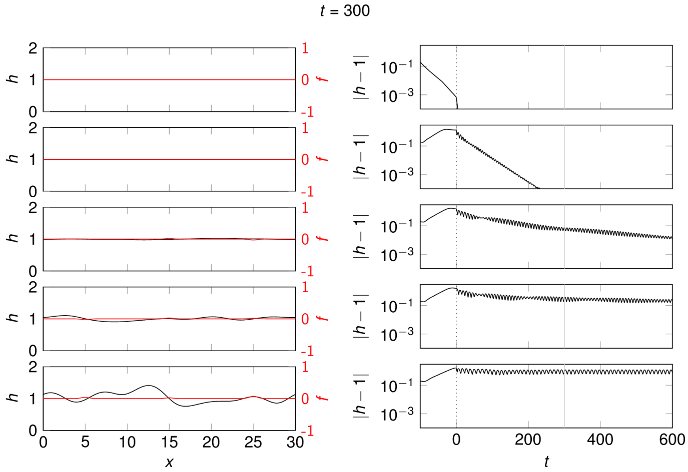

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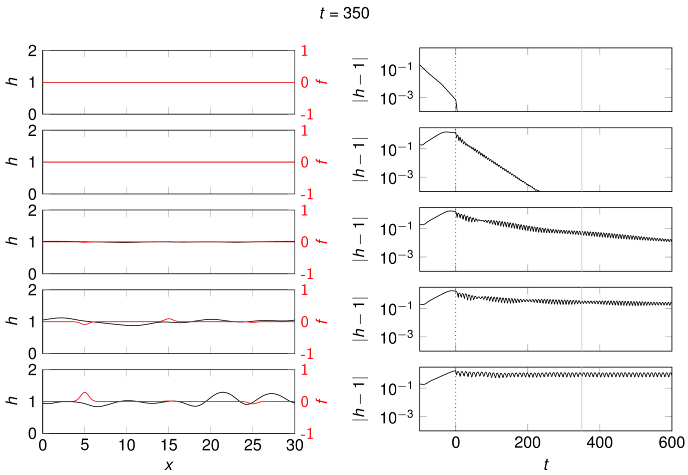

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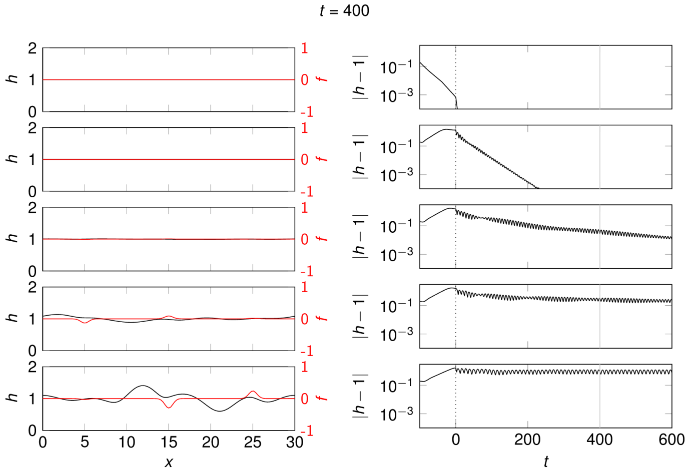

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Stability Analysis

What predictions can we make about the **stabilisability** of the system?

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Shifting to Fourier space and explicitly separating the unstable modes

$$\tilde{\xi}_t = \begin{bmatrix} \tilde{J}_u + \tilde{\Psi}_u \tilde{K}_u & 0 \\ \tilde{\Psi}_s \tilde{K}_u & \tilde{J}_s \end{bmatrix} \tilde{\xi}.$$

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So the **number of controls should exceed the number of unstable modes**.

Stability analysis

Unstable modes

If we look at a unimodal perturbation $h = 1 + \hat{h}e^{ikx+\lambda t}$ we have

$$\lambda^2 + \left(\frac{5}{2Re} + \frac{34}{21}ik\right)\lambda + \left(\frac{5}{Re}ik - \left[\frac{4}{7} - \frac{5\cot\theta}{3Re}\right]k^2 + \frac{5}{6Re\,Ca}k^4\right) = 0.$$

Stability analysis
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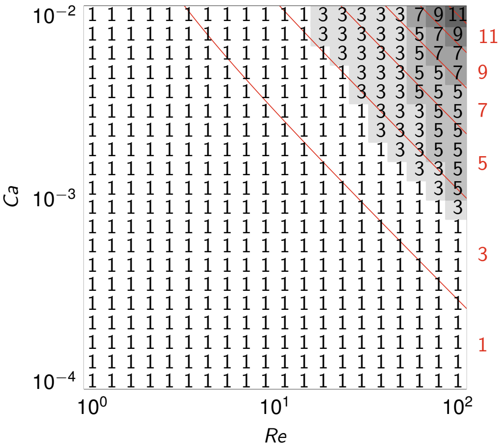
Rescaling to allow for $L \neq 2\pi$ we can compute the **unstable mode count**

$$n_{\mathrm{u}} = 1 + 2\left\lfloor \frac{L}{2\pi}\sqrt{Ca\left(\frac{8}{5}Re - 2\cot\theta\right)}\right\rfloor.$$

Stability analysis

Numerical comparison



In practice the controls **outperform the linear predictions**.

Conclusion

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Next steps:

- 3D flows
- Alternative actuator mechanisms
- Restricted observations
- Infinite-dimensional control

More detail

Preprint available on arXiv

arxiv.org/pdf/2301.11379

Code available on GitHub

github.com/OaHolroyd/falling-film-control/tree/paper-dec-2022